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Tractor calculus, BGG complexes, and the cohomology of cocompact Kleinian groups ☆

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ABSTRACT

For a compact, oriented, hyperbolic n-manifold (M, g), realised as $M = \Gamma\backslash\mathbb{H}^n$ where Γ is a torsion-free cocompact subgroup of $SO(n, 1)$, we establish and study a relationship between differential geometric cohomology on M and algebraic invariants of the group Γ. In particular for $\mathbb{F}$ an irreducible $SO(n, 1)$-module, we show that the group cohomology with coefficients $H^\bullet(\Gamma, \mathbb{F})$ arises from the cohomology of an appropriate projective BGG complex on M. This yields the geometric interpretation that $H^\bullet(\Gamma, \mathbb{F})$ parameterises solutions to certain distinguished natural PDEs of Riemannian geometry, modulo the range of suitable differential coboundary operators. Viewed in another direction, the construction shows one way that non-trivial cohomology can arise in a BGG complex, and sheds considerable light on its geometric meaning. We also use the tools developed to give a new proof that $H^1(\Gamma, S^k_0\mathbb{R}^{n+1}) \neq 0$ whenever M contains a compact, orientable, totally geodesic hypersurface. All constructions use another result that we establish, namely that the canonical flat connection on a hyperbolic manifold coincides with the tractor connection of projective differential geometry.

1. Introduction

Hyperbolic manifolds and their fundamental groups are objects that lie at the intersection of many different areas of mathematics. Indeed, their study has exposed deep interactions between analysis, group theory, Riemannian geometry, and topology, see for example [26,37,40,6]. In this article we show that there is considerable gain in introducing another tool to study hyperbolic manifolds; namely, *projective differential geometry*. We show that many standard objects in the area may be effectively interpreted in this framework. Critically, this perspective enables us to apply powerful tools such as the Cartan-tractor calculus

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and BGG theory [2,11,13,21]; these provide a route for the geometric interpretation of algebraic invariants of hyperbolic geometries. This approach is also important because there is obvious scope to adapt the ideas developed here for the treatment of (for example) locally symmetric spaces and, in the non-compact cases, their compactifications.

Projective differential geometry is based around geodesic structure. Two affine connections, ∇ and ∇', are said to be projectively related if they share the same geodesics as unparameterised curves. This happens trivially if connections differ only by torsion, so we take a *projective geometry* to mean a manifold equipped with an equivalence class $\boldsymbol{p} = [\nabla]$ of torsion-free affine connections which have the same geodesics up to parametrisation. In Riemannian geometry the metric canonically determines a distinguished affine connection, the Levi-Civita connection, and this provides a basic object for invariantly treating local geometric analysis. On projective geometries there is no such distinguished connection on the tangent bundle. However there is a canonical connection on a related higher rank bundle, called the *tractor connection*, and for projective geometry this is the basic tool for invariant local analysis.

Since hyperbolic manifolds have a Riemannian metric and Levi-Civita connection available, our reasons for exploiting the tractor connection are more subtle. The first point is that on a hyperbolic manifold the tractor connection is flat; indeed a hyperbolic manifold may be characterised as a flat projective manifold equipped with a certain Cartan holonomy reduction in the sense of [16,17]. In the hyperbolic geometry literature it is well known that the hyperbolic metric gives rise to another canonical flat connection, determined by the faithful representation of the fundamental group into the group of hyperbolic isometries. We show that this agrees with the tractor connection, and the identification of these connections is used to give a rich interplay between results in projective differential geometry and results for (higher dimensional) Kleinian groups. In particular, we use the projective tractor calculus and the related BGG theory (as discussed below) to study the cohomology, with coefficients, of a hyperbolic manifold. In the other direction, the construction shows one way that non-trivial cohomology can arise in a BGG complex, and sheds considerable light on its geometric meaning. This is significant because despite the recent growth of interest in geometric BGG sequences and complexes (following e.g. [11,21,23]) there has been little work on the determination and meaning of the BGG cohomology of parabolic geometries.

The work here also has motivation from another direction; namely, the general role of projective geometry in pseudo-Riemannian geometry. Clearly any pseudo-Riemannian geometry determines a projective structure through its Levi-Civita connection. However a deeper role of projective geometry in the metric setting has emerged [33,39], with already some striking developments and applications [10,27,32]. A particularly promising aspect is a strong link with certain overdetermined partial differential equations and their close connections to the invariant calculus for projective geometry [14,16,24]. The current work provides a new direction in this programme.

Since the work here brings together different areas of mathematics we have attempted to present the material in an elementary way and, to the extent possible, make the treatment self-contained.

Before discussing our results in more detail, let us point out that, as indicated above, there are two directions in which the work here should generalise in important ways. First compact hyperbolic manifolds form our main focus here, but much of the development applies equally to the case of infinite volume convex co-compact hyperbolic quotients; this should be especially interesting because of the relation to projective compactification defined in [14–16]. Second, many of the constructions in this article will generalise without difficulty to other classes of manifolds that arise by discrete quotients of homogeneous spaces. Thus natural extensions of the ideas developed here will allow analogous treatments of representation theory associated with locally symmetric spaces and, in the case of non-compact symmetric spaces, their compactifications. We view our treatment here as a template for the first aspects of these wider theories.