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A short guide through integration theorems of generalized distributions

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ABSTRACT

The generalization of Frobenius' theorem to foliations with singularities is usually attributed to Stefan and Sussmann, for their simultaneous discovery around 1973. However, their result is often referred to without caring much on the precise statement, as some sort of magic spell. This may be explained by the fact that the literature is not consensual on a unique formulation of the theorem, and because the history of the research leading to this result has been flawed by many claims that turned to be refuted some years later. This, together with the difficulty of doing proof-reading on this topic, brought much confusion about the precise statement of Stefan–Sussmann's theorem. This paper is dedicated to bring some light on this subject, by investigating the different statements and arguments that were put forward in geometric control theory between 1962 and 1994 regarding the problem of integrability of generalized distributions. We will present the genealogy of the main ideas and show that many mathematicians that were involved in this field made some mistakes that were successfully refuted. Moreover, we want to address the prominent influence of Hermann on this topic, as well as the fact that some statements of Stefan and Sussmann turned out to be wrong. In this paper, we intend to provide the reader with a deeper understanding of the problem of integrability of generalized distributions, and to reduce the confusion surrounding these difficult questions.

1. Introduction

Foliation theory is the study of foliations on manifolds. A *foliation* on a manifold M is a partition of M into connected immersed submanifolds, that are called *leaves*. A foliation is called *regular* if the leaves have the same dimension, and *singular* otherwise. Over every point $x \in M$, the tangent space of the leaf L_x through x is a subspace of the tangent space of M. The data of a subspace $\mathcal{D}_x$ of T_xM at every point $x \in M$ define what is called a *distribution* $\mathcal{D} = \bigcup_{x\in M} \mathcal{D}_x$ on M. Notice that a distribution is not necessarily a sub-bundle of TM because it may not have constant rank. For example, for a regular foliation, since the leaves have the same dimension, the induced distribution formed by the tangent spaces at every point has constant rank over M. In the singular case however, the dimension of the tangent spaces to the leaves may

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vary from leaf to leaf. Since the tangent spaces to a given foliation form a distribution $\mathcal{D}$ on M, and since the space of vector fields tangent to the leaves are closed under Lie bracket, then $\mathcal{D}$ inherits the Lie bracket of vector fields. More precisely, we say that a distribution $\mathcal{D}$ is *involutive* if for every two sections X, Y of $\mathcal{D}$, the commutator $[X, Y]$ is a section of $\mathcal{D}$ as well. On the other hand, a given distribution $\mathcal{D}$ may not come from the tangent spaces of a foliation. Then, we say that $\mathcal{D}$ is *integrable* if there exists a foliation such that each leaf L satisfies $T_xL = \mathcal{D}_x$ for every $x \in L$. A legitimate question is thus: 'Given a distribution on M, what are the conditions under which it is integrable to a foliation?'

This question is a modern formulation of a set of results and investigations that were related to – but not directly concerned with – the topic of integrating distributions into foliations. Originally, the problem emerged as finding the solutions of non-linear first-order partial differential equations, and was pioneered by Lagrange that provided a method for systems involving up to two independent variables. It was then formalized to an arbitrary number of variables by Pfaff in his memoir at the University of Berlin in 1815, hence the name of *Pfaffian systems* [25]. He showed how one may transform a set of n first-order non-linear partial differential equations into a set of $2n$ ordinary linear differential equations. The simplification method presented by Pfaff could be seen as finding a submanifold of the space of variables on which some specific one-form vanishes. The problem was that Pfaff could not make precise what were the conditions under which one could use this simplification. This question – designated as the *problem of Pfaff* – led to multiple investigations that finally found an accurate answer by Frobenius in 1877.

Actually, the name 'Frobenius' theorem' comes from Cartan in 1922 because Frobenius' result had an tremendous influence on Cartan's calculus of differential forms. Frobenius' paper is actually archetypal of the production of the Berlin school of Mathematics at that time, which promoted the idea that a clear, rigorous and systematic presentation of the arguments was just as important as the discovery of new results by whatever means. Frobenius and his contemporaries in Berlin participated in a shift of paradigm in modern mathematics by improving standards of rigor and presentation [25]. This is in part the reason why Frobenius is remembered for this theorem, whereas the work of his predecessors has been forgotten.

Indeed, it turns out that Frobenius' theorem is actually an algebraic reformulation of a result published in 1840 by Deahna [1], who then became a teacher in a secondary school (that was common at the time) before his premature death at age 28 in 1844. Deahna's work did not gain much interest, and it was later Clebsch in 1861, editing a posthumous article of Jacobi, who improved Pfaff's argument [2]. Even if the problem of solving Pfaffian systems had been around for many years, it was the article of Clebsch which motivated the interest of Frobenius on this question. Simultaneously, unaware of Clebsch's investigations, Natani proposed another approach to the question of solving Pfaffian systems, but the relationship with Clebsch's work was not realized before a few years [25]. The modern formulation of Frobenius' theorem does not correspond to the one appearing in its original paper [3], because it has been modified to fit with modern-day standards and conventions:

Theorem 1 *(Frobenius (1877)). Let M be a smooth manifold and let $\mathcal{D}$ be a smooth distribution of constant rank on M. Then $\mathcal{D}$ is integrable into a regular foliation if and only if $\mathcal{D}$ is involutive.*

Involutivity is a natural necessary condition because the set of vector fields on any leaf of a foliation is involutive, hence the corresponding distribution should be as well. Frobenius implicitly proved that it turns out to be a sufficient condition in the regular case, see [27] for a short proof.

Interestingly, the problem of integrating generalized distributions was approached in the same way as for the Frobenius' theorem: i.e. solving a set of linear differential equations. Indeed, at the turn of the 1950s, numerous investigations in the field of control theory − the study of the solvability of first-order differential equations under the influence of one or more external parameters − arose and developed in the following years. Unfortunately, the picture in control theory involves external parameters that modify and generalize