Differential Geometry and its Applications 60 (2018) 33–38

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Locally conformally Kähler manifolds with holomorphic Lee field

Andrei Moroianu [a], Sergiu Moroianu [b,1], Liviu Ornea [c,d,*,2]

[a] Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France
[b] Institutul de Matematică al Academiei Române, P.O. Box 1-764, RO-014700 Bucharest, Romania
[c] University of Bucharest, Faculty of Mathematics, 14 Academiei str., 70109 Bucharest, Romania
[d] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 21 Calea Grivitei, 010702-Bucharest, Romania

ARTICLE INFO

Article history:
Received 23 December 2017
Received in revised form 13 May 2018
Available online xxxx
Communicated by A. Swann

Dedicated to Professor Izu Vaisman at his eightieth anniversary

MSC:
53C55

Keywords:
lcK structures
Vaisman manifolds
Holomorphic Lee vector field

ABSTRACT

We prove that a compact lcK manifold with holomorphic Lee vector field is Vaisman provided that either the Lee field has constant norm or the metric is Gauduchon (i.e., the Lee field is divergence-free). We also give examples of compact lcK manifolds with holomorphic Lee vector field which are not Vaisman.

1. Introduction

Let (M, J, g) be a Hermitian manifold of complex dimension $n \geq 2$. Let ω be its fundamental form defined as $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$.

Definition. A Hermitian manifold (M, J, g) is called *locally conformally Kähler* (lcK) if there exists a closed 1-form θ which satisfies

$$d\omega = \theta \wedge \omega.$$

* Corresponding author.
E-mail addresses: andrei.moroianu@math.cnrs.fr (A. Moroianu), moroianu@alum.mit.edu (S. Moroianu), lornea@fmi.unibuc.ro, liviu.ornea@imar.ro (L. Ornea).

[1] Sergiu Moroianu was partially supported by the CNCS – UEFISCDI grant PN-III-P4-ID-PCE-2016-0330.
[2] Liviu Ornea was partially supported by the CNCS – UEFISCDI grant PN-III-P4-ID-PCE-2016-0065.

https://doi.org/10.1016/j.difgeo.2018.05.004
0926-2245/

If moreover θ is parallel with respect to the Levi-Civita connection of g, the manifold (and the metric itself) are called *Vaisman* [13].

We refer to [1] for definitions and main properties of lcK metrics.

Among lcK manifolds, Vaisman ones are important because their topology is better understood. Consequently, characterizations of this subclass are always interesting. Several sufficient conditions are known for a compact, non-Kähler lcK manifold to be Vaisman: the Einstein–Weyl condition ([12]); the existence of a parallel vector field ([7]); the pluricanonical condition ([8]); homogeneity ([3]); having potential and being embedded in a diagonal Hopf manifold ([9], [10]); or being toric ([4], [6]).

On the other hand, as compact Vaisman manifolds have zero Euler characteristic, the blow-up of a compact lcK manifold (which is known to be lcK, [14]) never admits Vaisman metrics; moreover, lcK Oeljeklaus–Toma manifolds (in particular, lcK Inoue surfaces) are never Vaisman, cf. e.g. [11].

It is known that on a Vaisman manifold, the Lee and anti-Lee vector fields $\theta^\sharp$ and $J\theta^\sharp$ are holomorphic and Killing. In this note, we discuss the implications of the Lee vector field being only holomorphic, and we add to the above list of sufficient conditions the following result:

Theorem 1. *Let (M, J, g, θ) be a compact locally conformally Kähler manifold with holomorphic Lee field $\theta^\sharp$. Suppose that one of the following conditions is satisfied:*

(i) *the norm of the Lee form θ is constant,* or
(ii) *the metric g is Gauduchon (which means by definition that the Lee form θ is co-closed with respect to g, see [2, pp. 502]).*

Then (M, J, g) is Vaisman.

As shown in Section 3, this result does not hold in general for holomorphic Lee vector fields $\theta^\sharp$ without imposing some additional hypotheses like (i) or (ii).

Very recently, Nicolina Istrati has found another instance where the conclusion of Theorem 1 holds:

Proposition *(Istrati [5]). Let (M, J, g, θ) be a compact locally conformally Kähler manifold with holomorphic Lee field. Suppose that the metric g has a potential, i.e., $\omega = \theta \wedge J\theta - dJ\theta$ (cf. [9]). Then (M, J, g) is Vaisman.*

2. Proof of Theorem 1

2.1. Local formulae

Consider the Lee vector field $T := \theta^\sharp$. We start by proving two straightforward results.

Lemma 2. *If T is holomorphic, then JT is both holomorphic and Killing.*

Proof. Since J is integrable and T is holomorphic, JT must also be holomorphic. On any lcK manifold, Cartan's formula yields:

$$\begin{aligned}\mathcal{L}_{JT}\,\omega &= d(JT \lrcorner\, \omega) + JT \lrcorner\, (\theta \wedge \omega) \\ &= -d\theta + (\theta \wedge \theta) = 0.\end{aligned}$$

Combined with $\mathcal{L}_{JT}\, J = 0$, this gives $\mathcal{L}_{JT}\, g = 0$. □