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Gradient estimates and Harnack inequalities for Yamabe-type parabolic equations on Riemannian manifolds

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A B S T R A C T

Let (M^n, g) be a complete noncompact n-dimensional Riemannian manifolds. In this paper, we consider the following Yamabe-type parabolic equation

$$u_t = \Delta u + au + bu^\alpha$$

on $M^n \times [0, \infty)$. We give a global gradient estimate of Hamilton-type for positive smooth solutions of this equation provided that Ricci curvature bounded from below. As its application, we show a dimension-free Harnack inequality and a Liouville-type theorem for nonlinear elliptic equations.

1. Introduction

Let (M^n, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. As a generalization of the Poincaré–Köbe Uniformization theorem, the Yamabe problem consists of finding a constant scalar curvature metric $\widetilde{g}$, which is pointwise conformally related to g. In 1984, Schoen [15] obtained a complete solution to the Yamabe problem on compact manifolds. This work combined new techniques with ideas developed in earlier work with Sing-Tung Yau, and partial results by Hidehiko Yamabe [18], Thierry Aubin [1] and Neil Trudinger [16]. The existence of a solution to the Yamabe problem is one of the greatest achievements of geometric analysis. Yamabe problem can be also formulated for other curvature quantities leading to interesting fully nonlinear or higher-order elliptic equations.

The conformal class of g is defined to be

$$[g] = \left\{ \widetilde{g} = u^{\frac{4}{n-2}} g : u \in C^\infty(M), u > 0 \right\}.$$

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If $\widetilde{g} = u^{\frac{4}{n-2}} g, u > 0$ we can compute the scalar curvature as

$$R_{\widetilde{g}} = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \left(\Delta_g u - \frac{n-2}{4(n-1)} R_g u \right).$$

Here R_g and $R_{\widetilde{g}}$ denote the scalar curvatures of g and g, respectively, and Δ_g is the Laplace–Beltrami operator associated with g. Thus the Yamabe problem amounts to finding a positive solution u of the familiar *Yamabe equation*

$$\Delta_g u - \frac{n-2}{4(n-1)} R_g u + \frac{n-2}{4(n-1)} K u^{\frac{n+2}{n-2}} = 0,$$

where $K = R_{\widetilde{g}}$ is constant.

A natural generalization of the classical Yamabe problem is the case where K is nonconstant and M is noncompact. When M is Euclidean space $\mathbb{R}^n$, Ni [12] considered the following nonlinear elliptic equation

$$\Delta u + K(x) u^{\frac{n+2}{n-2}} = 0, \tag{1.1}$$

where $K(x)$ is continuous function on M. If $K(x)$ is negative and decays slower than $C/|x|^2$ at ∞, Ni proved equation (1.1) has no positive solutions in M. In [2], Brandolini et al. considered the *Yamabe-type equation*

$$\Delta u(x) + a(x)u(x) + b(x)u^{\alpha}(x) = 0, \tag{1.2}$$

on a complete noncompact n-dimensional Riemannian manifold, where $a(x)$ and $b(x)$ are continuous functions and $\alpha > 1$. If $b(x) < 0$ every where, under some integrable conditions, the authors showed that (1.2) has no positive bounded solution. For further discussion on Yamabe's problem, we refer the reader to [6], [9], [10] and the references therein.

The nonlinear heat equation of the form

$$\left(\Delta - \frac{\partial}{\partial t} \right) u(x,t) + b(x,t) u^{\alpha}(x,t) = 0, \tag{1.3}$$

where α is a real constant, has attracted study of many mathematicians because its applications in mathematics, physics and many other fields. Equation (1.3) with $\alpha = 1$ is called the Schrödinger equation, which appears in quantum physics. When $\alpha < 0$, equation (1.3) can be viewed as a simple parabolic Lichnerowicz equation. It is well known that the Lichnerowicz equation arises from the Hamiltonian constraint equation for the Einstein-scalar field. In the case $\alpha > 1$, $b(x,t) = |x|^{\beta}$ with $\beta > -2$, equation (1.3) becomes Hénon parabolic equations. This equation was proposed by Hénon in [4] when he studied rotating stellar structures.

In 1986, Li and Yau [8] proved gradient estimates of the Schrödinger equation on complete Riemannian manifolds. By using the gradient estimates, Li and Yau obtained Harnack inequalities for positive solutions of Schrödinger equation; especially, they proved the optimal upper and lower bounds for the heat kernel. Later, Li–Yau's gradient estimates was investigated by many mathematicians. For related research and some improvements on Li–Yau's gradient estimates, we refer the reader to [5,13,14] and the references therein. When $0 < \alpha < \frac{n}{n-1}$, J. Li [7] proved gradient estimates of Li–Yau-type for the positive solution of (1.3) on Riemannian manifold with Ricci curvature bounded from below. As applications, he derived several parabolic Harnack inequalities. His results can be considered a generalization of the famous Li–Yau's gradient estimates (see [8]). In [22], Zhu introduced a Souplet–Zhang-type gradient estimate of (1.3) for $0 < \alpha < 1$ and obtained a Liouville-type theorem for bounded smooth solution of (1.3). Recently, in the case $\alpha < 1$, Wang et al. [17] considered gradient estimates of the Li–Xu-type for positive smooth solution of (1.3) on Riemannian manifold (M^n, g). Consequently, when the Ricci curvature is nonnegative they proved the Schrödinger-type equation