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The transitivity of geodesic flows on rank 1 manifolds without focal points

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ABSTRACT

In this article, we show that the geodesic flow on a compact rank 1 Riemannian manifold without focal points is transitive, which generalize the classical work of P. Eberlein in the case of nonpositive curvature.

1. Introduction and main results

In this article, we investigate the transitivity of geodesic flows on connected compact rank 1 manifolds without focal points.

Let (M, g) be a C^∞ connected compact n-dimensional Riemannian manifold, where g is a Riemannian metric. For any $p \in M$ and $v \in T_pM$, let γ_v be the unique geodesic satisfying the initial conditions $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Without special indication, all geodesics we are considering in this paper are the geodesics with unit speed. The geodesic flow on the unit tangent bundle SM is defined as:

$$\phi^t : SM \to SM, \qquad (p, v) \mapsto (\gamma_v(t), \gamma_v'(t)), \qquad \forall\, t \in \mathbb{R}.$$

Definition 1.1. Let γ be a geodesic on (M, g). Two distinct points $p = \gamma(t_1)$ and $q = \gamma(t_2)$ are called *focal* if there is a Jacobi field J along γ such that $J(t_1) = 0$, $J'(t_1) \neq 0$ and $\frac{d}{dt}\,|_{t=t_2}\, \|J(t)\|^2 = 0$. (M, g) is called a manifold *without focal points* if there is no focal points on any geodesic in (M, g).

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It is well known that nonpositively curved manifolds have no focal points. In [4], R. Gulliver construct examples of the manifolds without focal points, which have some regions with positive curvature. Thus manifolds without focal points can be regarded as a nontrivial generalization of the manifolds of nonpositive curvature.

Now we give the definition of the geometric rank, which measures the global flatness of (M, g) in a sense. This concept was first proposed by W. Ballmann–M. Brin–P. Eberlein in [1].

Definition 1.2. Let (M, g) be a complete manifold without focal points. For each $v \in SM$, we define $\text{rank}(v)$ to be the dimension of the vector space of the parallel Jacobi fields along the geodesic γ_v, and $\text{rank}(M) := \min\{\text{rank}(v) \mid v \in SM\}$. For a geodesic γ we define $\text{rank}(\gamma) = \text{rank}(\gamma'(t))$, $\forall\, t \in \mathbb{R}$.

We call (M, g) is a *rank* 1 *manifold* if $\text{rank}(M) = 1$. For example, manifolds of negative curvature are rank 1 manifolds. Recently, with F. Wang and W. Wu, we investigate the ergodicity and measures of maximal entropy for geodesic flows on rank 1 manifolds without focal points [5–7].

In [2], P. Eberlein proved that the geodesic flow on a compact rank 1 manifold of nonpositive curvature is *transitive*, i.e., there exists a unit tangent vector whose orbit (w.r.t. geodesic flow) is dense in SM. In this article, based on our previous results in [6], we generalize Eberlein's result to the manifold without focal points. This is the following theorem:

Theorem A. *Suppose (M, g) is a smooth compact rank 1 Riemannian manifold without focal points, then the geodesic flow $\phi^t : SM \to SM$ is transitive.*

The details of the proof of Theorem A will be given in the next section. There are some key differences between our proof and Eberlein's original proof. For example Eberlein's arguments use the law of cosines, which is no longer valid for the manifolds without focal points. We overcome the difficulties by using our earlier results in [6].

2. Proof of Theorem A

Let (M, g) be a compact Riemannian manifold without focal points, while $(X, \tilde{g})$ is its universal covering, and d is the distance on X induced by the lifted Riemannian metric $\tilde{g}$. It is a standard fact that $X/\Gamma = M$, where $\Gamma \approx \pi_1(M)$ is a discrete subgroup of the isometry group $\text{Iso}(X)$. In this paper, we use ϕ^t to represent both the geodesic flow on SM and the lifted geodesic flow on SX.

For two geodesics γ_1 and γ_2 in X, we call γ_1 and γ_2 are *positively asymptotic* (*negatively asymptotic*) if there is a positive number $C > 0$ such that

$$d(\gamma_1(t), \gamma_2(t)) \leq C, \quad \forall\, t \geq 0 \ (\forall\, t < 0).$$

The relation of positive (negative) asymptoticity is an equivalence relation among geodesics on X. The equivalence class of geodesics that are positively (negatively) asymptotic to a given geodesic γ is called *points at infinity*, which is denoted by $\gamma(+\infty)$ $(\gamma(-\infty))$. We use $X(\infty)$ to denote the set of all points at infinity.

Let $\overline{X} = X \cup X(\infty)$. For each point $p \in X$ and $v \in S_pX$, for any points $x \in \overline{X} - \{p\}$ and positive number ϵ, we define the following notations:

- $\gamma_{p,x}$ is the geodesic from p to x with $\gamma_{p,x}(0) = p$;
- $\measuredangle(v, x) = \measuredangle(v, \gamma'_{p,x}(0))$;
- $C(v, \epsilon) = \{q \in \overline{X} - \{p\} \mid \measuredangle(v, q) < \epsilon\}$.