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A note on twisted Dirac operators on closed surfaces

Volker Branding

University of Vienna, Faculty of Mathematics, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

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A B S T R A C T

We derive an inequality that relates nodal set and eigenvalues of a class of twisted Dirac operators on closed surfaces and point out how this inequality naturally arises as an eigenvalue estimate for the Spin^c Dirac operator. This allows us to obtain eigenvalue estimates for the twisted Dirac operator appearing in the context of Dirac-harmonic maps and their extensions, from which we also obtain several Liouville type results.

1. Introduction and results

Throughout this note we assume that (M, g) is a closed Riemannian spin surface with fixed spin structure. On the spinor bundle ΣM we have a metric connection $\nabla^{\Sigma M}$ induced from the Levi-Cevita connection and we fix a hermitian scalar product denoted by $\langle\cdot,\cdot\rangle_{\Sigma M}$. Sections of the spinor bundle are called *spinors*. In addition, we have the Clifford multiplication of spinors with tangent vectors, denoted by $X\cdot\psi$ for $X \in \Gamma(TM)$ and $\psi \in \Gamma(\Sigma M)$. Clifford multiplication is skew-symmetric

$$\langle X \cdot \psi, \xi\rangle_{\Sigma M} = -\langle \psi, X \cdot \xi\rangle_{\Sigma M}$$

and satisfies the Clifford relation

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi$$

for $X, Y \in \Gamma(TM)$ and $\psi, \xi \in \Gamma(\Sigma M)$.

E-mail address: volker.branding@univie.ac.at.

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The Dirac operator is a first order differential operator acting on spinors and is given by

$$D := e_1 \cdot \nabla^{\Sigma M}_{e_1} + e_2 \cdot \nabla^{\Sigma M}_{e_2},$$

where $\{e_1, e_2\}$ is an orthonormal basis of TM. It is an elliptic differential operator, which is self-adjoint with respect to the L^2-norm. On a compact Riemannian manifold the spectrum of the Dirac operator is discrete and consists of both positive and negative eigenvalues. In general, there are only a few manifolds on which the spectrum of the Dirac operator can be computed explicitly, see the book [12] for a current overview on the spectrum of the Dirac operator.

However, it is possible to estimate the spectrum. For instance, on a manifold of dimension $n \geq 3$, Friedrich [10] proved that all eigenvalues λ of the Dirac operator satisfy

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M R,$$

where R denotes the scalar curvature of the manifold. On closed surfaces Bär [1] proved that

$$\lambda^2 \geq \frac{2\pi\chi(M)}{\mathrm{Area}(M,g)},$$

where $\chi(M)$ is the Euler characteristic of M and $\mathrm{Area}(M,g)$ denotes the area of M.

Since $\dim M = 2$ and using the complex-volume form $\omega_{\mathbb{C}} = ie_1 \cdot e_2$, we can decompose the spinor bundle into its positive and negative parts, that is $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, where

$$\Sigma^{\pm} M = \frac{1}{2}(1 \pm \omega_{\mathbb{C}}) \cdot \Sigma M.$$

The Dirac type operators that appear in theoretical physics usually do not act on sections of ΣM but on sections of $\Sigma M \otimes E$, where E is a given vector bundle. We call $\Sigma M \otimes E$ a *twisted spinor bundle*. The connections on ΣM and E induce a connection on $\Sigma M \otimes E$ and the same holds true for the scalar product on $\Sigma M \otimes E$. Throughout this note we will always take the real part of the scalar product on $\Sigma M \otimes E$ turning it into a Euclidean scalar product. It will be denoted by $\langle \cdot, \cdot \rangle = \mathrm{Re}(\langle \cdot, \cdot \rangle_{\Sigma M \otimes E})$.

The *twisted Dirac operator* D^E maps sections of $\Sigma M \otimes E$ to sections of $\Sigma M \otimes E$ and is given by

$$D^E := e_1 \cdot \nabla^{\Sigma M \otimes E}_{e_1} + e_2 \cdot \nabla^{\Sigma M \otimes E}_{e_2}.$$

Note that the Clifford multiplication only acts on the first factor of $\Sigma M \otimes E$. The principal symbol of D^E can be computed as

$$\sigma_1(D^E) = \sigma_1(D, \eta) \otimes id_E, \qquad \eta \in \Gamma(T^*M),$$

hence this operator is still elliptic. Moreover, D^E is formally self-adjoint whenever we have a metric connection on E. The square of the twisted Dirac operator satisfies the following Weitzenböck formula [19, p. 164]

$$(D^E)^2 = \nabla^*\nabla^{\Sigma M \otimes E} + \frac{R}{4} + \frac{1}{2} \sum_{i,j=1}^{2} e_i \cdot e_j \cdot R^E(e_i, e_j), \tag{1.1}$$

where R^E denotes the curvature endomorphism of the twist bundle E.

For more background material on spin geometry and the Dirac operator we refer the reader to the books [11] and [19].