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Eigenvalue estimates of Reilly type in product manifolds and eigenvalue comparison for strip domains

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A B S T R A C T

In the first part we derive sharp upper bounds of Reilly type for three kinds of eigenvalues in product manifolds $\mathbb{R}^k \times M^{n+1-k}$ for any complete Riemannian manifold M. The eigenvalues include the first Laplacian eigenvalue on mean convex closed hypersurfaces, the first Steklov eigenvalue on domains with mean convex boundary, and the first Hodge Laplacian eigenvalue on closed hypersurfaces with certain convexity condition. In the second part, we prove a comparison result between the first Steklov eigenvalue of a strip domain in space forms and that of the corresponding warped product manifold.

1. Introduction

Eigenvalue estimates in terms of various geometric quantities have been among the most important topics in differential geometry. Of particular interest is the well-known Reilly inequality, which gives an extrinsic upper bound for the first nonzero eigenvalue $\lambda_1(\Sigma)$ of the Laplacian operator Δ_Σ on an isometric immersion of an m-dimensional closed Riemannian manifold (Σ^m, g) into the Euclidean space $\mathbb{R}^n$: ([14])

$$\lambda_1(\Sigma) \leq \frac{m}{|\Sigma|} \int_\Sigma |\vec{H}|^2 dv_g. \tag{1}$$

Here $|\Sigma|$ and dv_g stand for respectively the volume and the Riemannian volume element of (Σ, g), and $|\vec{H}|^2$ the square of the length of the mean curvature vector $\vec{H}$ of the immersion.

Since then, there have arisen all kinds of generalizations in different directions. For instance, the estimate above can be easily extended to immersed submanifolds in spheres, and after a partial result due to Heintze

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[8], the estimate in the same spirit has also proved valid for immersed submanifolds in hyperbolic spaces thanks to El Soufi and Ilias [3,4]. Moreover, eigenvalue estimates in terms of higher order mean curvatures have been obtained by Reilly himself [14] for Euclidean case and by Grosjean [6,7] for spherical and hyperbolic cases. No attempt is made here to survey the extensive works concerning this topic. Nevertheless, we would like to mention that most of the results after Reilly's are concerned with the submanifolds in space forms or in Riemannian manifolds with sectional curvature bounded from above or below.

Our first aim in this paper is to derive eigenvalue estimates of Reilly type in special Riemannian product manifolds $\mathbb{R}^k \times M^{n+1-k}$ for any complete Riemannian manifold M^{n+1-k}, with no condition on the sectional curvature of M. More precisely, we first give a sharp upper bound for the eigenvalue $\lambda_1(\Sigma)$ on a mean convex hypersurface Σ in $\mathbb{R} \times M^n$.

Theorem 1. *Let $x : \Sigma^n \to \mathbb{R} \times M^n$ be an oriented closed immersed hypersurface, where M^n is any complete Riemannian manifold. Moreover, assume that Σ is mean convex, i.e. $H > 0$. Then there holds:*

$$\lambda_1(\Sigma) \leq n\kappa_+(\Sigma)H_+(\Sigma),$$

where $\kappa_+(\Sigma)$ and $H_+(\Sigma)$ are the maximum principal curvature and maximum mean curvature of Σ, respectively.

Remark 2. In particular, if we view $\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$, then for sphere $\mathbb{S}^n(R) \subset \mathbb{R}^{n+1}$, we obtain

$$n\kappa_+(\mathbb{S}^n(R))H_+(\mathbb{S}^n(R)) = n\frac{1}{R^2}.$$

So we have an equality in this case, which shows the sharpness of our estimate.

Besides the Laplacian eigenvalue, the Steklov eigenvalue has also received more and more attention during the recent years. Let Ω be a domain with boundary $\Sigma = \partial\Omega$ in a Riemannian manifold $\tilde{M}$. The Steklov eigenvalue problem, introduced by V. A. Steklov in 1895 (see [11]), is

$$\begin{cases} \Delta_\Omega v = 0, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = pv, & \text{on } \Sigma, \end{cases} \tag{2}$$

where ν is the outward unit normal along Σ. The Steklov spectrum is a nonnegative, discrete and unbounded sequence:

$$0 = p_0(\Omega) < p_1(\Omega) \leq p_2(\Omega) \leq \cdots \nearrow \infty. \tag{3}$$

There is an extensive literature concerning the Steklov eigenvalue problem. We refer to the recent survey [5] and the references therein for an account of this topic. Among these known results, S. Ilias and O. Makhoul's [9] is most relevant to the present paper, where they proved Reilly-type estimates for the first nonzero Steklov eigenvalue on a compact submanifold with boundary in a Euclidean space or a sphere. In our product manifold setting, we can prove:

Theorem 3. *Let Ω be a domain in the product manifold $\mathbb{R} \times M^n$, where M^n is any complete Riemannian manifold. Moreover, assume the boundary $\Sigma = \partial\Omega$ is mean convex, i.e. $H > 0$. Then the first nonzero Steklov eigenvalue for Ω satisfies*

$$p_1(\Omega) \leq \kappa_+(\Sigma)\frac{H_+(\Sigma)}{H_-(\Sigma)},$$