Differential Geometry and its Applications 60 (2018) 116–131

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

On conformal pseudo-subriemannian fundamental graded Lie algebras of semisimple type

Tomoaki Yatsui

Masakae 1-9-2, Otaru 047-0003, Japan

ARTICLE INFO

Article history:
Received 19 November 2016
Available online xxxx
Communicated by D.V. Alekseevsky

MSC:
76M60

Keywords:
Fundamental graded Lie algebras
Conformal pseudo-subriemannian geometry

ABSTRACT

We introduce the notion of a conformal pseudo-subriemannian fundamental graded Lie algebra. Moreover we give a classification of conformal pseudo-subriemannian fundamental graded Lie algebras of semisimple type and their prolongations. Also we apply these results to a conformal pseudo-subriemannian geometry.
© 2018 Elsevier B.V. All rights reserved.

1. Introduction and notation

This paper is the sequel to the previous one [16]. We first recall the notion of fundamental graded Lie algebras. Moreover we define the notion of conformal pseudo-subriemannian fundamental graded Lie algebras, which is a generalization of conformal subriemannian fundamental graded Lie algebras.

A graded Lie algebra (GLA) $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is called a fundamental graded Lie algebra (FGLA) if it is a finite dimensional graded Lie algebra generated by nonzero subspace $\mathfrak{g}_{-1}$. An FGLA $\mathfrak{m}$ is said to be of the μ-th kind if $\mathfrak{g}_{-\mu} \neq \{0\}$ and $\mathfrak{g}_p = \{0\}$ for $p < -\mu$. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be an FGLA over $\mathbb{R}$ such that $\mathfrak{g}_{-2} \neq \{0\}$, and let $[g]$ be the conformal class of a nondegenerate symmetric bilinear form g on $\mathfrak{g}_{-1}$. Then the pair $(\mathfrak{m}, [g])$ is called a conformal pseudo-subriemannian FGLA. In particular if g is positive definite, then $(\mathfrak{m}, [g])$ is called a conformal subriemannian FGLA. Also if the signature of g has the form (r, r), then $(\mathfrak{m}, [g])$ is called a conformal neutral-subriemannian FGLA. Note that if $(\mathfrak{m}, [g])$ is a conformal pseudo-subriemannian FGLA, so is $(\mathfrak{m}, [-g])$. Given two conformal pseudo-subriemannian FGLAs $(\mathfrak{m}_1, [g_1])$ and $(\mathfrak{m}_2, [g_2])$ we say that $(\mathfrak{m}_1, [g_1])$ is isomorphic to $(\mathfrak{m}_2, [g_2])$ if there exists a graded Lie algebra isomorphism φ of $\mathfrak{m}_1$ onto $\mathfrak{m}_2$ such

E-mail address: yatsui@frontier.hokudai.ac.jp.

https://doi.org/10.1016/j.difgeo.2018.06.001
0926-2245/© 2018 Elsevier B.V. All rights reserved.

that $[\varphi^* g_2] = [g_1]$. Also we say that $(\mathfrak{m}_1, [g_1])$ is equivalent to $(\mathfrak{m}_2, [g_2])$ if $(\mathfrak{m}_1, [g_1])$ is isomorphic to $(\mathfrak{m}_2, [g_2])$ or $(\mathfrak{m}_2, [-g_2])$.

Let $(\mathfrak{m}, [g])$ be a conformal pseudo-subriemannian FGLA, and let $\mathfrak{g}_0$ be the Lie algebra consisting of all the derivations D of $\mathfrak{m}$ satisfying the following conditions: (1) $D(\mathfrak{g}_p) \subset \mathfrak{g}_p$ for all $p < 0$; (2) $D|\mathfrak{g}_{-1} \in \mathfrak{co}(\mathfrak{g}_{-1}, g)$. There exists a GLA $\mathfrak{l} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{l}_p$ such that: (i) $\mathfrak{g}_p = \mathfrak{l}_p$ for $p \leqq 0$; (ii) $\mathfrak{l} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{l}_p$ is transitive, i.e., for $X \in \mathfrak{l}_p$, $p \geqq 0$, if $[X, \mathfrak{l}_{-1}] = \{0\}$, then $X = 0$; (iii) $\mathfrak{l} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{l}_p$ is maximum among GLAs satisfying conditions (i) and (ii) above, which is called the prolongation of $(\mathfrak{m}, [g])$ (For more details on the prolongation, see [12, §5]). Note that the prolongation of $(\mathfrak{m}, [g])$ is finite dimensional (Lemma 3.2). Clearly the prolongation of $(\mathfrak{m}, [g])$ coincides with that of $(\mathfrak{m}, [-g])$.

It is known that the prolongation $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of a conformal subriemannian FGLA $(\mathfrak{m}, [g])$ satisfying the condition $\mathfrak{g}_1 \neq \{0\}$ is a real rank one simple graded Lie algebra ([4], [16]). In contrast, there exists a conformal neutral-subriemannian FGLA $(\mathfrak{m}, [g])$ such that the prolongation $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of $(\mathfrak{m}, [g])$ is nonsemisimple and such that $\mathfrak{g}_1 \neq \{0\}$ (cf. Example 5.1). A conformal pseudo-subriemannian FGLA is said to be of semisimple type if the prolongation is semisimple. In this paper we give a strong classification of conformal pseudo-subriemannian FGLAs of semisimple type and their prolongations (Theorem 5.2). Here, by a strong classification, we mean a classification using root systems plus an explicit realization of all the subspaces $\mathfrak{g}_p$. In particular we prove that the prolongation of a conformal pseudo-subriemannian FGLA of semisimple type is simple. In section 6 we give a local classification of maximally homogeneous conformal pseudo-subriemannian manifolds of semisimple type (Theorem 6.1).

Notation and conventions

(1) Blackboard bold is used for the standard systems $\mathbb{Z}$ (the ring of integers), $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{C}'$ (split complex numbers), the real division rings $\mathbb{H}$ (Hamilton's quaternions), $\mathbb{H}'$ (split quaternions), $\mathbb{O}$ (Cayley's [nonassociative] octonions) and $\mathbb{O}'$ (split octonions). We denote by $\mathbb{R}_{>0}$ (resp. $\mathbb{R}_{\geqq 0}$) the set consisting of all the positive real numbers (resp. non-negative real numbers). For $\mathbb{K} = \mathbb{C}$, $\mathbb{C}'$, $\mathbb{H}$, $\mathbb{H}'$, $\mathbb{O}$ or $\mathbb{O}'$, we set $\operatorname{Im}\mathbb{K} = \{\, z \in \mathbb{K} : \operatorname{Re} z = 0 \,\}$.

(2) For any real vector space V we denote by $V(\mathbb{C})$ the complexification of V.

(3) Let V be a finite dimensional real vector space, and let g be a nondegenerate symmetric bilinear form on V. We set

$$\mathfrak{so}(V, g) = \{\, A \in \mathfrak{gl}(V) : A \cdot g = 0 \,\},$$
$$\mathfrak{co}(V, g) = \{\, A \in \mathfrak{gl}(V) : A \cdot g = \eta_A g \text{ for some } \eta_A \in \mathbb{R} \,\},$$

where $A \cdot g$ is a symmetric bilinear form on V defined by $(A \cdot g)(x, y) = g(Ax, y) + g(x, Ay)$ $(x, y \in V)$. We define a linear mapping $g^\flat : V \to V^*$ by $g^\flat(x)(y) = g(x, y)$ $(x, y \in V)$. Since g is non-degenerate, $g^\flat$ is a linear isomorphism. We denote by $g^\sharp$ the inverse mapping of $g^\flat$.

(4) For a graded vector space $V = \bigoplus_{p \in \mathbb{Z}} V_p$ and $k \in \mathbb{Z}$ we denote subspaces $\bigoplus_{p \leqq k} V_p$ and $\bigoplus_{p \geqq k} V_p$ by $V_{\leqq k}$ and $V_{\geqq k}$ respectively. Also we denote the subspace $\bigoplus_{p<0} V_p$ by V_-. We call V_- the negative part of V.

(5) For a reductive Lie algebra $\mathfrak{g}$, we denote by $\mathfrak{g}^{ss}$ the semisimple part of $\mathfrak{g}$.

(6) For a GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ we denote by $\operatorname{Aut}_0(\mathfrak{g})$ the group consisting of all the automorphisms a of $\mathfrak{g}$ such that $a(\mathfrak{g}_p) = \mathfrak{g}_p$ for all $p \in \mathbb{Z}$.

Download English Version:

https://daneshyari.com/en/article/8898263

Download Persian Version:

https://daneshyari.com/article/8898263

Daneshyari.com