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Invariants for primary visual cortex

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A B S T R A C T

Metric and conformal structures, naturally related to primary visual cortex, and rational differential invariants of distributions are studied. The factor equations, describing orbits of regular distributions as well as their classification, are given.

1. Introduction

The classic experiments ([2], [3]) of David Hubel and Torsten Weisel published in 1959 (Gross Horwitz Prize, 1975, and Nobel Prize, 1981) gave us understanding how neurons extract information about light cast on the retina. They investigated how neurons in the primary visual cortex respond when they moved a bright contour in retina. They noticed that neurons react only if the line passed a particular place of retina and with a certain orientation. Moreover, "sometimes a moving spot gave more activation for one direction than for the opposite", ([3]).

Geometrically this result could be formulated in the following way. Assume that the retina is a 2-dimensional manifold M. Then for a given point of retina $a \in M$ "simple" neurons detect the point and an oriented line in the tangent plane, $p \subset T_a M$. Moreover, they constitute a "hypercolumn", which allows to detect any oriented line $p \subset T_a M$ at each point $a \in M$.

Oriented lines in tangent (or cotangent) planes to M (we'll work with cotangent planes) form so-called spherization of the cotangent bundle

$$\mathbf{S}(M) = (\mathbf{T}^* M \setminus 0) / \mathbb{R}^+$$

–the classical 3-dimensional contact manifold.

It is also known, that the simple neurons operate as filters (see [1], [6] for more details) on optic signal which could be considered as convolution with the Gaussian or Gabor filters. This means that, in addition to the contact structure, we have metric or, at least, conformal structure on M, (cf. [1,4,5,8]).

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In this paper we consider the following cases of surfaces M and the structure groups G, defining the following geometrical structures:

- Metric structures:
  1. Plane Euclidean geometry, $M = \mathbb{R}^2$ and $G = \mathbf{SE}(2)$ – the special Euclidean group of rigid motions of the plane, and
  2. Spherical geometry, $M = \mathbf{S}^2$ and $G = \mathbf{SO}(3, \mathbb{R})$-the special orthogonal group of rigid motions of the sphere.
- Conformal structure:
  $M = \mathbf{S}^2 = \mathbb{C}\mathbf{P}^1$ and $G = \mathbf{SL}(2, \mathbb{C})$ – the group of Mobius transformations.

Together with oriented lines we'll consider also the case of nonoriented lines. In this case, instead of spherization $\mathbf{S}(M)$, we get the projectivization of the cotangent bundle:

$$\mathbf{P}(M) = (\mathbf{T}^*M \setminus 0) / \mathbb{R}^*,$$

where $\mathbb{R}^* = \mathbb{R} \setminus 0$ is the group of nonzero real numbers.

Remark that any section $w : M \to \mathbf{S}(M)$ (or $w : M \to \mathbf{P}(M)$) of the spherical (or projective) bundle $\sigma : \mathbf{S}(M) \to M$ (or $\pi : \mathbf{P}(M) \to M$) could be considered as an oriented (or nonoriented) distributions on M. Integral curves of these distribution one could interpreted as intensity levels of light ([1]).

Assuming that the neurons as filters do not change under the action of the structure group G we describe invariants of the distributions under this action. In all these cases Lie–Tresse theorem ([10]) could be applied and it shows that the field of rational invariants is generated by some basic invariants and invariant derivations.

Moreover, this field separates regular orbits of (jets of) oriented distributions and gives us criteria for recognition of regular distributions with respect to different structure groups.

The structure of the invariant field shows that G-orbits of regular distributions correspond to solutions of some systems of differential equations (we call them factor equations, ([10])). In the given case, we've found explicit forms for this equations. All of them are so-called Lax type equations and very probably posses some kind of integrability.

2. Actions

We consider distributions on surface M, which are defined by differential 1-forms $\omega \in \Omega^1(M) : a \in M \mapsto \operatorname{Ker} \omega_a \subset T_aM$. Two 1-forms ω_1 and ω_2 define the same nonoriented distribution if and only if $\omega_2 = f\omega_1$, for some function f without zeroes, and $f > 0$, for the case of oriented distributions.

In other words, oriented distributions could be considered as sections of the spheric bundle $\sigma : \mathbf{S}(M) \to M$, but nonoriented distributions are sections of the projectivization of the cotangent bundle $\pi : \mathbf{P}(M) \to M$.

Consider differential 1-forms as sections of the cotangent bundle $\tau^* : \mathbf{T}^*M \to M$, then the action of vector fields on differential 1-forms on M give us the Hamiltonian lift $X \mapsto \widehat{X}$ of vector fields on M to Hamiltonian vector fields on $\mathbf{T}^*M$. These lifts commute with the $\mathbb{R}^*$-action on $\mathbf{T}^*M$ and therefore induces vector fields on $\mathbf{S}(M)$ and $\mathbf{P}(M)$.

Namely, for given local coordinates (x, y) on M and the corresponding canonical coordinates (x, y, p_1, p_2) on $\mathbf{T}^*(M)$, any vector field $X = a(x, y)\partial_x + b(x, y)\partial_y$ on M lifts on the cotangent bundle $\mathbf{T}^*M$ by taking the Hamiltonian vector field X_H with Hamiltonian $H = a(x, y)p_1 + b(x, y)p_2$.

It is easy to check that this lift gives a morphism of the Lie algebra of vector fields on M to the Lie algebra of Hamiltonian vector fields on $\mathbf{T}^*M$.