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Why are normal sub-Riemannian extremals locally minimizing?

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It is well-known that normal extremals in sub-Riemannian geometry are curves that locally minimize the length functional (equivalently, the energy functional). Most proofs of this fact do not make, however, an explicit use of relations between local optimality and the geometry of the problem. In this paper, we provide a new proof of that classical result, which gives insight into direct geometric reasons for local optimality. Also the relation of the regularity of normal extremals with their optimality becomes apparent in our approach.

1. Introduction

Motivations Our initial motivation to undertake this research comes from our recent study [7]. In that paper we developed a contact geometry approach to optimal control problems. As a particular application we discussed sub-Riemannian (*SR*, in short) geodesic problems and obtained elegant geometric characterizations of both normal and abnormal SR extremals. In particular, *normal SR extremals* (*NSRE*s, in short) can be nicely described in terms of the distribution orthogonal to the actual extremal and the flow related with the optimal control. That result, here formulated as Theorem 2.2, was first obtained in [2]. Clearly, NSREs satisfy only first-order conditions for optimality, yet it is well-known that these suffice for their local optimality. Thus it is natural to ask the following question:

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How does the geometry of NSREs reflect their local optimality? (Q)

In this paper we answer question (Q) providing a detailed geometric proof of local optimality of NSREs (Theorem 2.3). Our original idea, which allowed to relate the geometric characterization of NSREs with their local optimality, is to study homotopies (and the related variations) of SR trajectories. We discuss the details of our approach in the next paragraph.

The standard proofs of the local optimality of NSREs (see, e.g., [8,9]) are deeply rooted in symplectic geometry. Usually one starts with the Hamiltonian description of a NSRE γ provided by the Pontryagin Maximum Principle [10]. The basic idea is to construct a solution of a Hamilton–Jacobi equation in a neighborhood of γ, and then to use this solution to build the so-called *calibration* of γ, i.e., a closed 1-form which estimates the SR length from below with an equality on γ (see Sec. 1.9 of [9] and for another proof using similar arguments [11]). To prove the local optimality of the considered NSRE we integrate the calibration over a closed contour containing γ and use Stokes theorem. That method is very elegant and powerful although it does not use explicitly the geometric reasons behind the local optimality of NSREs. We believe that the new proof presented in this work, using homotopies, gives an insight into these reasons. For example, the relation between optimality and regularity for NSREs is apparent in our approach. In our proof we do not use symplectic geometry, working directly with the SR distribution and the SR metric, and use only basic differential-geometric tools. We hope that similar ideas will allow to study the optimality issues for other classes of optimal control problems. Let us note that another point of view on NSREs can be found in [4], where local optimality of smooth extremals (for a general optimal control problem) is discussed from the perspective of the second-order necessary optimality conditions.

Outline of the proof Let us now sketch the main steps of the proof and explain geometric reasons which make NSREs locally optimal.

First observe that since we are interested in local optimality only, without any loss of generality we can reformulate the (local) SR geodesic problem as an optimal control problem on $\mathbb{R}^n$, quadratic in cost and linear in controls.

Naively, to prove that a given NSRE $\gamma_0 : [0, T] \to \mathbb{R}^n$ (corresponding to a control $u(t)$) is optimal, one should show that any other trajectory of the system sharing the same end-points as γ_0 has energy (length) bigger than that of γ_0. It is, however, quite difficult to describe SR trajectories with given end-points, so instead let us consider any SR trajectory $\gamma_1 : [0, T] \to \mathbb{R}^n$ (corresponding to a control $u(t) + \Delta u(t)$) for which we assume only that it has the same initial point as $\gamma_0(\cdot)$ and that the energy of $\gamma_1(\cdot)$ is smaller than the energy of $\gamma_0(\cdot)$. Showing that the end-point $\gamma_1(T)$ is necessarily different from $\gamma_0(T)$ will end the proof.

In order to compare these end-points, we extend $\gamma_0(\cdot)$ and $\gamma_1(\cdot)$ to a natural homotopy $\gamma_s : [0, T] \to \mathbb{R}^n$ with $s \in [0, 1]$, simply by considering SR trajectories corresponding to the intermediate controls $u(t)+s\Delta u(t)$ and all sharing the same initial point $\gamma_0(0)$. Now we shall concentrate our attention on the variation of the family $\gamma_s(\cdot)$ with respect to the parameter s:

$$b_s(t) := \partial_s \gamma_s(t) \ .$$

It is clear that the vector $b_0(T)$ approximates at $\gamma_0(T)$ the curve of end-points $\gamma_s(T)$ joining $\gamma_0(T)$ and $\gamma_1(T)$.

It turns out that it is easy to obtain an analytic formula for $b_0(T)$, see Lemma 3.2, and that the geometric characterization of NSREs (Theorem 2.2) yields a specific behavior of this vector – it has to point "much" backward with respect to $\dot{\gamma}_0(T)$ – see Fig. 1. In other words, the fact that a given SR trajectory $\gamma_0(\cdot)$ is a NSRE implies that the end-points $\gamma_s(T)$ of the natural homotopy joining it with $\gamma_1(\cdot)$ has to wind backward along $\gamma_0(\cdot)$ (provided that the energy of $\gamma_1(\cdot)$ is not greater than that of $\gamma_0(\cdot)$). If T is sufficiently small, then the curve $\gamma_s(T)$ has "no time" to return to $\gamma_0(T)$, enforcing $\gamma_0(T) \neq \gamma_1(T)$ and thus proving the optimality of $\gamma_0(\cdot)$ and, in particular, uniqueness.