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An integral formula for a class of biharmonic maps from Euclidean 3-space [☆]

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A R T I C L E I N F O

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A B S T R A C T

We classify biharmonic semi-conformal maps from domains of Euclidean 3-space whose fibres are arcs of circles in terms of holomorphic data. The geometry behind this classification enables us to give an integral formula which produces biharmonic maps which are now no longer necessarily semi-conformal. An integral invariant for bounded singular sets of more general semi-conformal maps is obtained.

1. Introduction

Let (M, g) and (N, h) be smooth Riemannian manifolds with M compact. A smooth map $\varphi : M \to N$ is called harmonic if it is a critical point of the energy functional

$$E_1(\varphi) = \frac{1}{2}\int_M ||\mathrm{d}\varphi||^2 \mathrm{d}v_g,$$

where $||\mathrm{d}\varphi||$ is the Hilbert–Schmidt norm of φ. Equivalently, φ is harmonic if it satisfies the Euler–Lagrange equations

$$\tau(\varphi) = \operatorname{trace} \nabla \mathrm{d}\varphi = 0\,,$$

which can be used to define a harmonic map more generally when the domain is no longer compact. A map $\varphi : (M, g) \to (N, h)$ is called biharmonic if it is a critical point of the bi-energy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \mathrm{d}v_g.$$

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The Euler–Lagrange equations for this functional are given by

$$\tau_2(\varphi) := -\mathrm{trace}\,(\nabla^\varphi)^2\tau(\varphi) - \mathrm{trace}\,R^N(\tau(\varphi), \mathrm{d}\varphi)\mathrm{d}\varphi = 0\,,$$

where ∇^φ is the connection in the pull-back bundle $\varphi^* TN$ and, if $\{e_i\}$ is a local orthonormal frame field on M, then

$$\mathrm{trace}\,(\nabla^\varphi)^2\tau(\varphi) = \left(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} - \nabla_{\nabla_{e_i}e_i}\right)\tau(\varphi)\,,$$

where here and subsequently, we sum over repeated indices. In particular, if φ is defined from an open subset of $\mathbb{R}^3$ into $\mathbb{C}$, then φ is biharmonic if and only if $\Delta(\Delta\varphi) = 0$, where Δ is the standard Laplacian. On the other hand, φ is semi-conformal if and only if

$$\sum_{i=1}^{3}\left(\frac{\partial\varphi}{\partial x_i}\right)^2 = 0,$$

where $(x_i)_{i=1,2,3}$ are standard coordinates on $\mathbb{R}^3$. It is well-known that a semi-conformal map from an open subset $V \subset \mathbb{R}^3$ is harmonic (and so a harmonic morphism) if and only if its regular fibres are straight lines, and furthermore such a mapping $z = \varphi(x)$ has a characterization in terms of holomorphic data [5], a so-called Weierstrass representation, given by

$$\xi_1(z)x_1 + \xi_2(z)x_2 + \xi_3(z)x_3 = 1, \tag{1}$$

where $(\xi_1, \xi_2, \xi_3) : \mathbb{C} \to \mathbb{C}^3 \setminus \{(0, 0, 0)\}$ is a holomorphic *null curve* meaning that

$$\xi_1(z)^2 + \xi_2(z)^2 + \xi_3(z)^2 = 0\,.$$

Note that such curves arise in the representation of minimal surfaces in $\mathbb{R}^3$, see [10] for a discussion in the context of spectral geometry.

Provided neither ξ_3 nor $\xi_1 - i\xi_2$ is identically zero, any such triple can be written $(1 - g(z)^2, i(1 + g(z)^2), 2g(z))/2h(z)$ for meromorphic functions g and h with zeros and poles suitably arranged, whereby (1) takes the form:

$$(1 - g(z)^2)x_1 + i(1 + g(z)^2)x_2 + 2g(z)x_3 = 2h(z)\,. \tag{2}$$

It is now a remarkable fact that the Whittaker formula for a harmonic function follows by contour integration:

$$\varphi(x_1, x_2, x_3) = \oint \psi\big((1 - z^2)x_1 + i(1 + z^2)x_2 + 2zx_3, z\big)\mathrm{d}z\,,$$

where ψ is a meromorphic function of two complex variables [13]. In fact, the general expression of a harmonic function on a domain of Euclidean 3-space is more subtle that this formula would suggest; a point we discuss more fully in §4.

A semi-conformal map $z = \varphi(x)$ from an open subset of $\mathbb{R}^3$ into $\mathbb{C}$ whose fibres are arcs of circles is locally given by an equation

$$x.x + 2\xi(z)\cdot x + \xi(z)^2 = 0, \tag{3}$$

where $\xi(z) = (\xi_1(z), \xi_2(z), \xi_3(z))$ is a triple of holomorphic functions and $\xi^2(z) = {\xi_1}^2(z) + {\xi_2}^2(z) + {\xi_3}^2(z)$ [1]. On replacing ξ by η/η^2, this becomes