Differential Geometry and its Applications 59 (2018) 1–11

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Euler–Arnold equations and Teichmüller theory

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A R T I C L E I N F O

Article history:
Received 1 September 2016
Received in revised form 18 January 2018
Available online xxxx
Communicated by T. Ratiu

MSC:
35Q35
53D25

Keywords:
Euler–Weil–Petersson equation
Global existence
Universal Teichmüller space
Sobolev metrics of fractional order

A B S T R A C T

In this paper we prove that for $s > 3/2$, all H^s solutions of the Euler–Weil–Petersson equation, which describes geodesics on the universal Teichmüller space under the Weil–Petersson metric, will remain in H^s for all time. This extends the work of Escher–Kolev for strong Riemannian metrics to the borderline case of $H^{3/2}$ metrics. In addition we show that all H^s solutions of the Wunsch equation, a variation of the Constantin–Lax–Majda equation which also describes geodesics on the universal Teichmüller curve under the Velling–Kirillov metric, must blow up in finite time due to wave breaking, extending work of Castro–Córdoba and Bauer–Kolev–Preston. Finally we illustrate these phenomena numerically.

1. Introduction

Euler–Arnold equations are PDEs that describe the evolution of a velocity field for which the Lagrangian flow is a geodesic in a group of smooth diffeomorphisms of a manifold, for some choice of right-invariant Riemannian metric; see Arnold–Khesin [1]. In the one-dimensional case, we will consider the diffeomorphism group of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. If the Riemannian metric is defined at the identity by

$$\langle u, u\rangle_r = \int_{S^1} u\Lambda^{2r} u \, d\theta, \tag{1}$$

where Λ^{2r} is a symmetric, positive pseudodifferential operator of order r, we call it a Sobolev H^r metric, and the Euler–Arnold equation is given by

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1 S.C. Preston gratefully acknowledges support from Simons Collaboration Grant no. 318969, and the hospitality of the organizers of the "Math on the Rocks" workshop in Grundsund, Sweden, where part of this research was conducted.

https://doi.org/10.1016/j.difgeo.2018.03.001
0926-2245/

$$m_t + um_\theta + 2mu_\theta = 0, \qquad m = \Lambda^{2r}u, \qquad u = u(t,\theta), \qquad u(0) = u_0 \in C^\infty(S^1). \tag{2}$$

Special cases include the Camassa–Holm equation when $r = 1$ and $\Lambda^2 = 1 - \partial_\theta^2$, or the right-invariant Burgers' equation when $r = 0$ and $\Lambda^0 = 1$ [4]. One can also allow Λ^{2r} to be degenerate (nonnegative rather than positive); the best known example is when $r = 1$ and $\Lambda^2 = -\partial_\theta^2$, for which we get the Hunter–Saxton equation [12]. Here we are interested in the fractional order cases $r = \frac{1}{2}$ and $r = \frac{3}{2}$ (see Escher–Kolev [7]), which arise naturally in Teichmüller theory [10]. Both cases are critical in some sense, due to the Sobolev embedding being critical: for $r < \frac{1}{2}$ Lagrangian trajectories do not depend smoothly on initial conditions, while for $r > \frac{3}{2}$ conservation of energy is strong enough to ensure global existence [8]. In this paper we will show that all solutions for $r = \frac{1}{2}$ blow up in finite time while for $r = \frac{3}{2}$ all smooth solutions exist globally; previously only some solutions were known to blow up in the $r = \frac{1}{2}$ case [2] and smooth solutions were only known to stay in $H^{3/2}$ in the $r = \frac{3}{2}$ case [10].

Specifically the cases we are interested in are

- ($r = \frac{1}{2}$) the Wunsch equation [22], [2]: $\Lambda^1 = Hu_\theta$,
- and ($r = \frac{3}{2}$) the Euler–Weil–Petersson equation [10]: $\Lambda^3 = -H(u_{\theta\theta\theta} + u_\theta)$,

where H is the Hilbert transform defined for periodic functions by $H(e^{in\theta}) = -i \operatorname{sign} n e^{in\theta}$. The Wunsch equation is a special case of the modified Constantin–Lax–Majda equation [16] which models vorticity growth in an ideal fluid.

When paired with the flow equation

$$\frac{\partial \eta}{\partial t}(t,\theta) = u\big(t, \eta(t,\theta)\big), \qquad \eta(0,\theta) = \theta, \tag{3}$$

the Euler–Arnold equation (2) describes geodesics $\eta(t)$ of the right-invariant Riemannian metric defined at the identity element by (1) on the homogeneous space $\mathrm{Diff}(S^1)/G$. Here G is the group generated by the subalgebra $\ker\Lambda$ of length-zero directions: for the Euler–Weil–Petersson equation we have $G = \mathrm{PSL}_2(\mathbb{R})$, and for the Wunsch equation we have $G = \mathrm{Rot}(S^1) \cong S^1$.

The local existence result was obtained by Escher–Kolev [7], a strengthening of a result of Escher–Kolev–Wunsch [9].

Theorem 1 *(Escher–Kolev). Suppose Λ^r is either $\Lambda^1 = Hu_\theta$ or $\Lambda^3 = -H(u_{\theta\theta\theta}+u_\theta)$. Then the system (2)–(3) is a smooth ODE for $\eta \in \mathrm{Diff}^s(S^1)/G$, for $s > \frac{3}{2}$ and $G = \mathrm{Rot}(S^1)$ or $G = \mathrm{PSL}_2(\mathbb{R})$, respectively. Hence for any $u_0 \in H^s(S^1)$, there is a unique solution $\eta\colon [0,T) \to \mathrm{Diff}^s(S^1)/G$ with $\eta(0) = \mathrm{id}$ and $\eta_t(0) = u_0$, with the map $u_0 \mapsto \eta(t)$ depending smoothly on u_0.*

Loss of smoothness in the equation (2) occurs due to the fact that composition required to get $u = \dot\eta \circ \eta^{-1}$ is not smooth in η; thus although the second-order equation for η (with u eliminated) is an ODE, the first-order equation (2) for u alone is *not* an ODE. This approach to the Euler equations was originally due to Ebin–Marsden [6]; for the Wunsch equation it was proved by Escher–Kolev–Wunsch [9] for large Sobolev indices, while for the Euler–Weil–Petersson equation it was proved by Escher–Kolev [7]. Castro–Córdoba [3] showed that if u_0 is initially odd, then solutions to the Wunsch equation blow up in finite time; the authors of [2] extended this result to some data without odd symmetry. For the Euler–Weil–Petersson equation, it was not known whether initially smooth data would remain smooth for all time. However Gay-Balmaz and Ratiu [10] interpreted the equation in $H^{3/2}$ as a *strong* Riemannian metric on a certain manifold and concluded that the velocity field u remains in $H^{3/2}(S^1)$ for all time. We complement this to obtain a uniform C^1 bound, which then by bootstrapping gives uniform bounds on all Sobolev norms H^s for $s > \frac{3}{2}$, and thus in particular we show that initially smooth solutions remain smooth.