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Euler–Arnold equations and Teichmüller theory

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ABSTRACT

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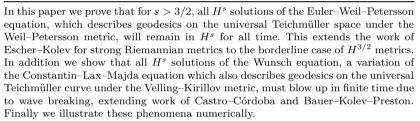
1. Introduction

Euler–Arnold equations are PDEs that describe the evolution of a velocity field for which the Lagrangian flow is a geodesic in a group of smooth diffeomorphisms of a manifold, for some choice of right-invariant Riemannian metric; see Arnold–Khesin [1]. In the one-dimensional case, we will consider the diffeomorphism group of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. If the Riemannian metric is defined at the identity by

$$\langle u, u \rangle_r = \int\limits_{S^1} u \Lambda^{2r} u \, d\theta, \tag{1}$$

where Λ^{2r} is a symmetric, positive pseudodifferential operator of order r, we call it a Sobolev H^r metric, and the Euler–Arnold equation is given by

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DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

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$$m_t + um_\theta + 2mu_\theta = 0, \qquad m = \Lambda^{2r} u, \qquad u = u(t,\theta), \qquad u(0) = u_0 \in C^{\infty}(S^1).$$
 (2)

Special cases include the Camassa-Holm equation when r = 1 and $\Lambda^2 = 1 - \partial_{\theta}^2$, or the right-invariant Burgers' equation when r = 0 and $\Lambda^0 = 1$ [4]. One can also allow Λ^{2r} to be degenerate (nonnegative rather than positive); the best known example is when r = 1 and $\Lambda^2 = -\partial_{\theta}^2$, for which we get the Hunter–Saxton equation [12]. Here we are interested in the fractional order cases $r = \frac{1}{2}$ and $r = \frac{3}{2}$ (see Escher–Kolev [7]), which arise naturally in Teichmüller theory [10]. Both cases are critical in some sense, due to the Sobolev embedding being critical: for $r < \frac{1}{2}$ Lagrangian trajectories do not depend smoothly on initial conditions, while for $r > \frac{3}{2}$ conservation of energy is strong enough to ensure global existence [8]. In this paper we will show that all solutions for $r = \frac{1}{2}$ blow up in finite time while for $r = \frac{3}{2}$ all smooth solutions exist globally; previously only some solutions were known to blow up in the $r = \frac{1}{2}$ case [2] and smooth solutions were only known to stay in $H^{3/2}$ in the $r = \frac{3}{2}$ case [10].

Specifically the cases we are interested in are

- $(r = \frac{1}{2})$ the Wunsch equation [22], [2]: $\Lambda^1 = Hu_{\theta}$,
- and $(r = \frac{3}{2})$ the Euler–Weil–Petersson equation [10]: $\Lambda^3 = -H(u_{\theta\theta\theta} + u_{\theta})$,

where H is the Hilbert transform defined for periodic functions by $H(e^{in\theta}) = -i \operatorname{sign} n e^{in\theta}$. The Wunsch equation is a special case of the modified Constantin–Lax–Majda equation [16] which models vorticity growth in an ideal fluid.

When paired with the flow equation

$$\frac{\partial \eta}{\partial t}(t,\theta) = u(t,\eta(t,\theta)), \qquad \eta(0,\theta) = \theta, \tag{3}$$

the Euler–Arnold equation (2) describes geodesics $\eta(t)$ of the right-invariant Riemannian metric defined at the identity element by (1) on the homogeneous space $\text{Diff}(S^1)/G$. Here G is the group generated by the subalgebra ker Λ of length-zero directions: for the Euler–Weil–Petersson equation we have $G = \text{PSL}_2(\mathbb{R})$, and for the Wunsch equation we have $G = \text{Rot}(S^1) \cong S^1$.

The local existence result was obtained by Escher–Kolev [7], a strengthening of a result of Escher–Kolev– Wunsch [9].

Theorem 1 (Escher–Kolev). Suppose Λ^r is either $\Lambda^1 = Hu_\theta$ or $\Lambda^3 = -H(u_{\theta\theta\theta} + u_\theta)$. Then the system (2)–(3) is a smooth ODE for $\eta \in \text{Diff}^s(S^1)/G$, for $s > \frac{3}{2}$ and $G = \text{Rot}(S^1)$ or $G = \text{PSL}_2(\mathbb{R})$, respectively. Hence for any $u_0 \in H^s(S^1)$, there is a unique solution $\eta \colon [0,T) \to \text{Diff}^s(S^1)/G$ with $\eta(0) = \text{id}$ and $\eta_t(0) = u_0$, with the map $u_0 \mapsto \eta(t)$ depending smoothly on u_0 .

Loss of smoothness in the equation (2) occurs due to the fact that composition required to get $u = \dot{\eta} \circ \eta^{-1}$ is not smooth in η ; thus although the second-order equation for η (with u eliminated) is an ODE, the first-order equation (2) for u alone is *not* an ODE. This approach to the Euler equations was originally due to Ebin–Marsden [6]; for the Wunsch equation it was proved by Escher–Kolev–Wunsch [9] for large Sobolev indices, while for the Euler–Weil–Petersson equation it was proved by Escher–Kolev [7]. Castro–Córdoba [3] showed that if u_0 is initially odd, then solutions to the Wunsch equation blow up in finite time; the authors of [2] extended this result to some data without odd symmetry. For the Euler–Weil–Petersson equation, it was not known whether initially smooth data would remain smooth for all time. However Gay-Balmaz and Ratiu [10] interpreted the equation in $H^{3/2}(S^1)$ for all time. We complement this to obtain a uniform C^1 bound, which then by bootstrapping gives uniform bounds on all Sobolev norms H^s for $s > \frac{3}{2}$, and thus in particular we show that initially smooth solutions remain smooth. Download English Version:

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