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Classification of homogeneous almost α-coKähler three-manifolds

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ABSTRACT

An orientable Riemannian three-manifold (M, g) admits an almost α-coKähler structure with g as a compatible metric if and only if M admits a foliation, defined by a unit closed 1-form, of constant mean curvature. Then, we show that a simply connected homogeneous almost α-coKähler three-manifold is either a Riemannian product of type $\mathbb{R} \times \mathbb{S}^2(k^2)$, equipped with its standard coKähler structure, or it is a semidirect product Lie group $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$ equipped with a left invariant almost α-coKähler structure. Moreover, we distinguish the several spaces of this classification by using the Gaussian curvature K_G of the canonical foliation. In particular, $\mathbb{R} \times \mathbb{S}^2(k^2)$ is the only simply connected homogeneous almost α-coKähler three-manifolds whose canonical foliation has Gaussian curvature $K_G > 0$.

1. Introduction

Almost coKähler manifolds were introduced and first studied by Blair [1], cf. also Goldberg and Yano [10], by involving an almost contact metric structure, under the name of "almost cosymplectic manifolds".

A cosymplectic manifold, in the wider but related sense, as originally defined by Libermann [15,16], is a smooth $(2n+1)$-manifold admitting a closed 1-form η and a closed 2-form ω, such that $\eta \wedge \omega^n$ is a volume form. If the 1-form η and the 2-form ω are not closed, but $\eta \wedge \omega^n$ is a volume form, then (M, η, ω) is called an *almost cosymplectic manifold*. We note that *any almost cosymplectic structure* (η, ω) *on* M *admits an associated almost contact metric structure* (φ, ξ, η, g) *with the same* η *and whose fundamental* 2*-form is given by* ω. If (η, ω) is cosymplectic, an associated almost contact metric structure (φ, ξ, η, g) is said to be an *almost coKähler structure*. If in addition the almost contact structure is normal, then M is called

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a *coKähler manifold*. Since the work by Li [14], in several papers (see, for example, [4], [6], [9]) "almost cosymplectic manifolds"(in the sense of [1], [10] and [19]) are called *almost* coKähler manifolds. We refer to the survey [5] for a overview on cosymplectic geometry and its relation with other areas of mathematics (in particular with the geometric mechanics) as well as with physics.

An almost cosymplectic structure (η, ω) on an odd-dimensional manifold M is said to be *locally conformally cosymplectic* if for any point $p \in M$ there exists an open neighborhood U of p and a smooth function $\sigma : U \to \mathbb{R}$, such that $e^{-\sigma}\eta|_U$ and $e^{-2\sigma}\omega|_U$ are closed (cf. also Olszak [20] and Vaisman [23]).

In [4] an almost cosymplectic structure (η, ω) is said to be *α-cosymplectic* if η is closed and $d\omega = 2\alpha\eta \wedge \omega$ for some $\alpha \in \mathbb{R}$. Any α-cosymplectic structure is locally conformally cosymplectic, and a cosymplectic structure is α-cosymplectic with $\alpha = 0$. In the same paper, the authors characterize cosymplectic and α-cosymplectic Lie algebras in terms of corresponding symplectic Lie algebras and suitable derivations on them. Moreover, [4] contains classification results in dimension five for cosymplectic, K-cosymplectic and coKähler Lie algebras.

Now, let (η, ω) be an α-cosymplectic structure on M and let (φ, ξ, η, g) be an associated almost contact metric structure. In such a case, the almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be an *almost α-coKähler manifold*. So, by this definition, we can treat almost coKähler and almost α-Kenmotsu structures in a unified way ([4], [12]).

The main purpose of this paper is to give a complete classification of simply connected homogeneous almost α-coKähler three-manifolds.

In Section 2 we give some basic information on almost cosymplectic and almost coKähler manifolds.

In Section 3 we study the geometry of an arbitrary almost α-coKähler three-manifold, in particular we characterize the coKähler case in terms of curvature, besides we study some properties related to the Gaussian and the extrinsic curvatures of the canonical foliation of an almost α-coKähler three-manifold. Then, we show that an orientable Riemannian three-manifold (M, g) admits an almost α-coKähler structure with g as a compatible metric if and only if M admits a foliation, defined by a unit closed 1-form, of constant mean curvature (see Theorem 3.1).

In Section 4 we describe explicitly several examples of almost α-coKähler structures on three-dimensional Lie groups.

In Section 5, we then show that a simply connected homogeneous almost α-coKähler three-manifold is either a Riemannian product of type $\mathbb{R} \times \mathbb{S}^2(k^2)$, equipped with its standard coKähler structure, where $\mathbb{S}^2(k^2)$ is a sphere of constant curvature $k^2 > 0$, or it is a semidirect product Lie group $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$ equipped with a left invariant almost α-coKähler structure. Moreover, we distinguish the several spaces of this classification by using the Gaussian curvature K_G of the canonical foliation (see Theorem 5.1 and Theorem 5.2). In particular, a simply connected homogeneous almost α-coKähler three-manifold whose canonical foliation has Gaussian curvature $K_G > 0$ (resp. $K_G < 0$) is $\mathbb{R} \times \mathbb{S}^2(k^2)$ equipped with its standard coKähler structure (resp. is isomorphic to the Lie group $\mathbb{R} \times \mathbb{H}^2(-k^2)$).

2. Preliminaries

2.1. Almost cosymplectic structures

We report below some basic information on almost cosymplectic structures.

Definition 2.1. *An* almost cosymplectic structure *on a smooth manifold M of odd dimension $2n+1$ is a pair (η, ω), where η is a 1-form and ω is a 2-form, such that $\eta \wedge \omega^n$ is a volume form on M. If $\omega = d\eta$, then (M, η) is a contact manifold. When $d\eta = 0$ and $d\omega = 0$, the structure is said to be* cosymplectic.