Differential Geometry and its Applications 59 (2018) 91–111

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Left-symmetric bialgebroids and their corresponding Manin triples ☆

Jiefeng Liu [a], Yunhe Sheng [b,*], Chengming Bai [c]

[a] Department of Mathematics, Xinyang Normal University, Xinyang 464000, Henan, China
[b] Department of Mathematics, Jilin University, Changchun 130012, Jilin, China
[c] Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, China

ARTICLE INFO

Article history:
Received 17 July 2017
Received in revised form 16 April 2018
Available online 25 April 2018
Communicated by T. Ratiu

MSC:
17B62
53D12
53D17
53D18

Keywords:
Left-symmetric algebroid
Left-symmetric bialgebroid
Pseudo-Hessian manifold
Pre-symplectic algebroid
Manin triple

ABSTRACT

In this paper, we introduce the notion of a left-symmetric bialgebroid as a geometric generalization of a left-symmetric bialgebra and construct a left-symmetric bialgebroid from a pseudo-Hessian manifold. We also introduce the notion of a Manin triple for left-symmetric algebroids, which is equivalent to a left-symmetric bialgebroid. The corresponding double structure is a pre-symplectic algebroid rather than a left-symmetric algebroid. In particular, we establish a relation between Maurer–Cartan type equations and Dirac structures of the pre-symplectic algebroid which is the corresponding double structure for a left-symmetric bialgebroid.

1. Introduction

Left-symmetric algebras (or pre-Lie algebras) arose from the study of convex homogeneous cones [28], affine manifolds and affine structures on Lie groups [10], deformation and cohomology theory of associative algebras [6] and then appeared in many fields in mathematics and mathematical physics. See the survey article [2] and the references therein. In particular, there are close relations between left-symmetric algebras and certain important left-invariant structures on Lie groups like the aforementioned affine, symplectic, Kähler, and metric structures [8,11,18,20,21]. A quadratic left-symmetric algebra is a left-symmetric alge-

☆ This research is supported by NSF of China (11471139, 11425104), NSF of Jilin Province (20170101050JC) and Nanhu Scholars Program for Young Scholars of XYNU.
* Corresponding author.
E-mail address: shengyh@jlu.edu.cn (Y. Sheng).

https://doi.org/10.1016/j.difgeo.2018.04.003
0926-2245/

bra together with a nondegenerate invariant skew-symmetric bilinear form [3]. A symplectic (Frobenius) Lie algebra is a Lie algebra $\mathfrak{g}$ equipped with a nondegenerate 2-cocycle $\omega \in \wedge^2\mathfrak{g}^*$. There is a one-to-one correspondence between symplectic (Frobenius) Lie algebras and quadratic left-symmetric algebras.

A left-symmetric algebroid, also called a Koszul–Vinberg algebroid, is a geometric generalization of a left-symmetric algebra. See [12,23,24] for more details and applications. In [13], we introduced the notion of a pre-symplectic algebroid, which is a geometric generalization of a quadratic left-symmetric algebra. Generalizing the relation between symplectic (Frobenius) Lie algebras and quadratic left-symmetric algebras, we showed that there is a one-to-one correspondence between symplectic Lie algebroids and pre-symplectic algebroids. See [4,7,19,22] for more details about symplectic Lie algebroids and their applications.

The purpose of this paper is to study the bialgebra theory for left-symmetric algebroids and the corresponding Manin triple theory. Motivated by [14,16], we introduce the notion of a left-symmetric bialgebroid, which is a geometric generalization of a left-symmetric bialgebra [1]. The double of a left-symmetric bialgebroid is not a left-symmetric algebroid anymore, but a pre-symplectic algebroid. This result is parallel to the fact that the double of a Lie bialgebroid,[1] is not a Lie algebroid, but a Courant algebroid [14]. Furthermore, if we consider the commutator of a left-symmetric bialgebroid, we obtain a matched pair of Lie algebroids, whose double is the symplectic Lie algebroid associated to the pre-symplectic algebroid. The above results can be summarized into the following commutative diagram:

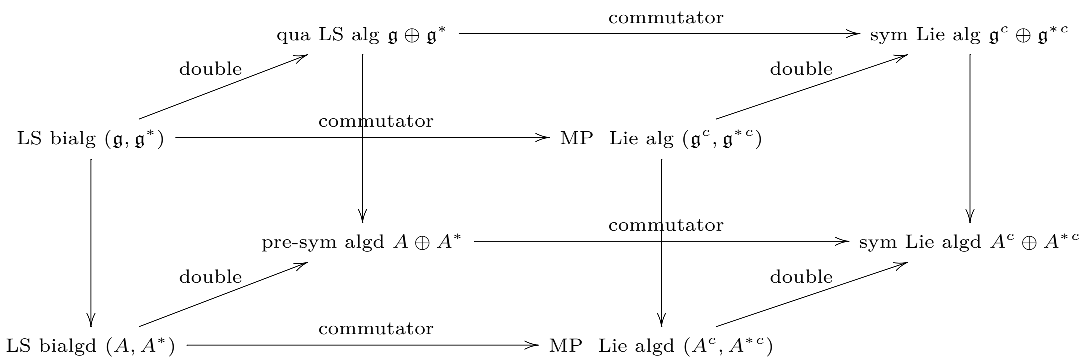

Diagram I

In the above diagram, qua is short for quadratic, LS is short for left-symmetric, alg is short for algebra, sym is short for symplectic, MP is short for matched pair and algd is short for algebroid. We establish a relation between left-symmetric bialgebroids and pseudo-Hessian manifolds. See [25–27] for more information about pseudo-Hessian Lie algebras and Hessian geometry. A flat manifold (M, ∇) gives rise to a left-symmetric algebroid $T_\nabla M$. We show that a pseudo-Riemannian metric g on (M, ∇) is a pseudo-Hessian metric if $\delta g = 0$, where δ is the cohomology operator of the left-symmetric algebroid $T_\nabla M$ (Proposition 4.13). Given a pseudo-Hessian manifold (M, ∇, g), $(T_\nabla M, T^*_H M)$ is a left-symmetric bialgebroid (Proposition 4.14), where H is the inverse of g. This result is parallel to that $(TM, T^*_\pi M)$ being a Lie bialgebroid for any Poisson manifold (M, π) [9,16]. It seems therefore that our theory is a symmetric analogue of Poisson geometry.

The paper is organized as follows. In Section 2, we give a review on Lie algebroids, left-symmetric algebroids and pre-symplectic algebroids. In Section 3, we develop the differential calculus on a left-symmetric algebroid which is the main tool in our later study. In Section 4, we introduce the notion of a left-symmetric bialgebroid and study its properties. In Section 5, we introduce the notion of a Manin triple for left-symmetric algebroids and show the equivalence between left-symmetric bialgebroids and Manin triples for left-symmetric algebroids.

Throughout this paper, all the vector bundles are over the same manifold M.

[1] The notion of a Lie bialgebroid was first introduced by Mackenzie and Xu as the infinitesimal object of a Poisson groupoid [16].