Differential Geometry and its Applications 59 (2018) 138–153

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Invariant Einstein metrics on certain compact semisimple Lie groups

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ARTICLE INFO

Article history:
Received 27 February 2018
Available online xxxx
Communicated by D.V. Alekseevsky

MSC:
53C25
53C35
53C30

Keywords:
Compact semisimple Lie groups
Einstein metrics
Naturally reductive metrics
Standard homogeneous Einstein manifolds

ABSTRACT

In this paper, we study left invariant Einstein metrics on compact semisimple Lie groups. A new method to construct holonomy irreducible non-naturally reductive Einstein metrics on certain compact semisimple (non-simple) Lie groups is presented. In particular, we show that if G is a classical compact simple Lie group and H is a closed subgroup such that G/H is a standard homogeneous Einstein manifold, then there exist holonomy irreducible non-naturally reductive Einstein metrics on $H \times G$, except for some very special cases. A further interesting result of this paper is that for any compact simple Lie group G, there always exist holonomy irreducible non-naturally reductive Einstein metrics on the compact semisimple Lie groups G^n, for any $n \geq 4$.

1. Introduction

The goal of this article is to present a new method to construct homogeneous Einstein manifolds which are non-naturally reductive and holonomy irreducible. Recall that a connected Riemannian manifold (M, g) is called Einstein if there exists a constant μ such that $\mathrm{Ric}(g) = \mu g$, where $\mathrm{Ric}(g)$ is the Ricci tensor of (M, g). Einstein metrics have many applications to physics as well as to other scientific fields, hence their study has always been one of the central problems in differential geometry. However, the problems in this field are generally rather involved. For example, till now a sufficient and necessary condition for a manifold to admit an Einstein metric is still unknown. Another remarkable long standing problem is whether there

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[1] Z. Yan is supported by NSFC (no. 11401425, 11626134, 11701300) and K.C. Wong Magna Fund in Ningbo University.
[2] S. Deng is supported by NSFC (no. 11671212, 51535008) of China.

https://doi.org/10.1016/j.difgeo.2018.04.005
0926-2245/

is a nonstandard Einstein metric on the 4-sphere S^4, see for example [28]. This problem particularly reveals the fact that it is essential to find new examples of Einstein metrics.

In the homogeneous case the study is relatively more fruitful and many beautiful results have been established. However, a complete classification of homogeneous Einstein manifolds still seems to be unreachable. Even if in the compact case, the classification has only been achieved for symmetric spaces, normal homogeneous spaces and naturally reductive metrics, see [8,14,30,32]. See also [3–5,7,9,26,31] for some important and interesting results on the existence (or non-existence) of homogeneous or inhomogeneous Einstein metrics on some special manifolds. Meanwhile, in the literature there are some excellent surveys of the development of this field, see for example [2,20,25,29].

It is somehow surprising that left invariant Einstein metrics on compact semisimple Lie groups have not been widely studied. This problem was first considered by G. Jensen [12], and then by J.E. D'Atri and W. Ziller [8]. It follows from their works that every compact simple Lie group except SO(3) admits a left invariant Einstein metric different from the bi-invariant metric. Note that their examples of Einstein metrics on compact semisimple Lie groups are naturally reductive, see also [11,13,17,21] for the study of naturally reductive Einstein metrics. The problem of finding left invariant Einstein metrics on compact Lie groups which are not naturally reductive is more difficult, and is stressed by J.E. D'Atri and W. Ziller in [8]. In 1994, Mori [16] initiated the study of this problem by showing that there exist non-naturally reductive Einstein metrics on the Lie group $\mathrm{SU}(n)$ with $n \geq 6$, using the method of Riemannian submersions. Later, in [1], the authors established the existence of non-naturally reductive Einstein metrics on the compact simple Lie groups $\mathrm{SO}(n)$ with $n \geq 11$, $\mathrm{Sp}(n)$ with $n \geq 3$, and the exceptional groups E_6, E_7 and E_8. Recently, some non-naturally reductive Einstein metrics have been found on the compact simple Lie groups $\mathrm{SO}(n)$ with $n = 5, n \geq 7$, G_2 and F_4, see [6,11,27]. The above results show that, except for SU(2), SU(3), SU(4), and SU(5), every compact simple Lie group admits left invariant non-naturally reductive Einstein metrics.

Recall that a Riemannian manifold (M, g) is called holonomy irreducible if the holonomy group is irreducible as a linear group acting on the tangent space (see [15], vol. 1, p. 179). It is well known that any connected simply connected holonomy reducible Einstein manifold can be decomposed into a direct product of holonomy irreducible Einstein manifolds. Now it is interesting to study the problem conversely, namely, whether a holonomy reducible Einstein manifold admits a holonomy irreducible Einstein metric or not.

In particular, we consider the following

Problem. *Does a compact non-simple semisimple Lie group admit a non-naturally reductive Einstein metric which is holonomy irreducible?*

Up to now, all known holonomy irreducible left invariant Einstein metric on compact non-simple semisimple Lie groups are naturally reductive, and are constructed by a Riemannian submersion

$$\mathrm{diag} : G \to G^n = \overbrace{G \times G \times \cdots \times G}^{n} \to G^n/\mathrm{diag}(G),$$

where G is compact simple and the first imbedding is diagonal, see Theorem 2.3 in Section 2.

In [27], we presented a method to construct a large number of left invariant non-naturally reductive Einstein metrics on certain compact simple Lie groups. In this paper, we follow this method, and use the results of the classification of standard homogeneous Einstein manifolds to construct infinite many holonomy irreducible non-naturally reductive Einstein metrics on some compact non-simple semisimple Lie groups.

The main idea of our method can be described as follows. We start with a compact connected standard homogeneous Einstein manifold $(G/H, g_{B_G})$ [30], where G and H are compact connected semisimple Lie groups. Assume that $B_H = cB_G|_{\mathfrak{h}}$, $0 < c < 1$, where B_H, B_G are the negative of Killing forms of the Lie algebras $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{g} = \mathrm{Lie}(G)$ respectively. For $\alpha > 0$, let $b = B_H + \alpha B_G$ and consider a certain