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On non-±holomorphic conformal minimal two-spheres in a complex Grassmannian $G(2,5)$ with constant curvature

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ABSTRACT

We classify the linearly full non-±holomorphic conformal minimal two-spheres with constant curvature and isotropy order no less than 2 in $G(2,5)$ up to $U(5)$. We also classify non-±holomorphic strongly isotropic linearly full conformal minimal two-spheres which are reducible at both sides in $G(2,n)$.

1. Introduction

It is an interesting problem in differential geometry to classify the minimal submanifolds in homogeneous spaces up to actions of their isometry groups. Canonical homogeneous spaces are sphere S^n and complex projective space $\mathbb{C}P^{n-1}$. Let ψ be a minimal immersion from S^2 into S^{2n}. We assume that it does not lie in any equator of S^{2n}. If the curvature of ψ is constant, Calabi proved that up to $SO(2n+1)$, ψ is constructed by spherical harmonics [7]. For $\mathbb{C}P^{n-1}$, there is a similar result. Let φ be a minimal immersion from S^2 to $\mathbb{C}P^{n-1}$. Similarly, we assume that it does not lie in any linear subspace of $\mathbb{C}^n$. In [12], J. Eells and J. Wood proved that φ was generated from a holomorphic map by taking derivatives and orthogonalization. Moreover, if φ is of constant curvature, Bolton et al. [3] proved that, up to $U(n)$, φ is a Veronese map. The proof is based on the Calabi's rigidity theorem: a linearly full Kähler submanifold in $\mathbb{C}P^{n-1}$ is determined up to $U(n)$ by its metric [6]. Calabi's original proof is to construct a distance function in the sense of metric space by Kähler potential. Then he proved Kähler submanifolds with same pull-back metric tensor

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are isometric under this distance function. So the axioms of Euclid imply they are congruent with respect to $U(n)$. Later, Griffiths gave another proof for holomorphic maps from Riemann surfaces by the theory of moving frame [14, Section 4].

Regarding $\mathbb{C}P^{n-1}$ as the complex Grassmannian $G(1,n)$, a natural idea is to extend these results to minimal immersions from Riemann surface to a complex Grassmannian $G(k,n)$. In [10], Chi and Zheng classified holomorphic two-spheres of constant curvature 2 in $G(2,4)$ into two families, and they are not equivalent to each other. So the Calabi's rigidity theorem does not hold for a complex Grassmannian. A natural question is to classify the minimal two-spheres of constant curvature.

Many people made progress on this problem. In 1999, Li and Yu [22] classified holomorphic two-spheres with constant curvature in $G(2,4)$ into four families, and non-±holomorphic minimal two-spheres into four cases. In 2004, Jiao and Peng [19] classified nonsingular holomorphic two-spheres with constant curvature in $G(2,5)$ with degree less than 5. In 2015, He, Jiao and Zhou [17] classified totally unramified linearly full holomorphic two-spheres with constant curvature in $G(2,5)$.

So in this paper, we study the classification of conformal minimal two-spheres with constant curvature in $G(2,5)$ which are not holomorphic and anti-holomorphic. For convenience, we call them *non-±holomorphic*. We hope our results together with works of Jiao and Peng [19], and He, Jiao and Zhou [17] would give a relatively complete understanding of minimal two-spheres with constant curvature in $G(2,5)$. As Li and Yu's results [22] in $G(2,4)$, there are non-homogeneous holomorphic two-spheres with continuous parameters [17, 19], while non-±holomorphic minimal two-spheres are "fewer" than holomorphic case. The known examples are direct sum of the Veronese maps. In the future research, we will try to understand this phenomenon.

Our approach is as follows: Let $\varphi : S^2 \to G(2,5)$ be a minimal immersion with constant curvature. We identity φ with the pull-back subbundle $\underline{\varphi}$ of the trivial bundle $S^2 \times \mathbb{C}^5$. The fiber of $\underline{\varphi}$ at $x \in S^2$ is $\underline{\varphi}_x = \varphi(x)$. If φ is holomorphic, then the composition of φ with *Plücker imbedding* is a Veronese map [10]. If the isotropy order of φ is finite or φ is reducible at one side, then we can split $\underline{\varphi}$ into two line bundles. In any case, the rigidity theorem of the Veronese maps can still be used. In particular, to study minimal two-sphere which is reducible at both sides, we give propositions about the relative positions between Veronese maps and use them to verify our results.

By [24, Theorem 3.6], the harmonic sequence of φ terminates after finite steps. So we describe all the possible maps by their isotropy order and rank. For each case, we split the trivial bundle $\underline{\mathbb{C}}^5$ into line bundles. They are perpendicular to each other, and their local sections form a unitary frame. The Maurer–Cartan forms of φ give the information about metric and Kähler angle. In general, these 1-forms are too complex to calculate. Because they have both $(1,0)$ and $(0,1)$ parts. By the theory of harmonic sequence, we can choose the orthogonal decomposition carefully. It guarantees that the Maurer–Cartan forms are either $(1,0)$ forms or $(0,1)$ forms. Then we can do some calculations. Moreover, some line bundles correspond to conformal immersions into complex projective spaces, and they are well studied. Finally, using the theory of absolute value type function developed by Li and Yu [22], we verify our classification theorems.

We state the main results of this paper:

Theorem 1.1. *Let $\varphi : S^2 \to G(2,n)$ be a linearly full non-±holomorphic strongly isotropic conformal minimal two-spheres in $G(2,n)$ with constant curvature K_φ. If φ is reducible at both sides, then up to $U(n)$, φ is equivalent to one of the following minimal immersions (the integer i below is fixed):*

(1) $K_\varphi = \frac{4}{n-2}$, $\underline{\varphi} = \underline{V_i^i} \oplus \underline{V_0^{n-i-2}}$, *for some* $i = 1, \dots, n-3$,
(2) $K_\varphi = \frac{4}{n-2+2i(n-2-i)}$, $\underline{\varphi} = \underline{V_i^{n-2}} \oplus \underline{\mathbb{C}v}$, *where v is a non-zero constant vector, for some* $i = 1, \dots, n-3$,
(3) $K_\varphi = \frac{4}{2(n-2)+i(2n-4-2i)}$, $\underline{\varphi} = \underline{V_i^{n-1}} \oplus \underline{V_{i+1}^{n-1}}$, *for some* $i = 1, \dots, n-3$,

where $\{\underline{V}_i^m, 0 \le i \le m\}$ is the Veronese sequence in $\mathbb{C}P^m$.