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# Conformal Killing–Yano 2-forms

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ABSTRACT

Riemannian manifolds carrying 2-forms satisfying the Killing–Yano equation and the conformal Killing–Yano equation are natural generalizations of nearly Kähler and Sasakian manifolds. In this article we exhibit new solutions of these equations. We also provide obstructions for their existence on Lie groups, and reduce the study of conformal Killing–Yano 2-forms to a particular class of non degenerate Killing–Yano 2-forms.



## 1. Introduction

A $p$-form $\omega$ on a Riemannian manifold $(M, g)$ is called *Killing–Yano* if it satisfies the equation

$$(\nabla_X \omega)(Y, X_1, \dots, X_{p-1}) = -(\nabla_Y \omega)(X, X_1, \dots, X_{p-1}) \tag{1}$$

for any vector fields $X, Y, X_1, \dots, X_{p-1}$ on $M$. In the case of $p = 1$, a Killing–Yano 1-form is dual to a Killing vector field. In this sense, Killing–Yano forms are natural generalizations of Killing vector fields. They were first introduced by K. Yano ([15]), who showed that Killing–Yano forms give rise to quadratic first integrals of the geodesic equation. This was first used by R. Penrose and M. Walker ([13]) to integrate the equation of motion.

In [12] the Killing–Yano equation is studied for the fundamental forms defining a $G$-structure, for $G = SO(n)$, $SU(n)$, $U(n)$, $Sp(n) \times Sp(1), Sp(n), G_2$ or $Spin(7)$. He proves that if the fundamental form satisfies (1) then, in most cases, it is parallel with respect to the Levi-Civita connection. The case of a compact simply connected symmetric space $M$ has been considered in [5] where it is shown that $M$ carries a non-parallel Killing–Yano $p$-form, $p \geq 2$, if and only if it is isometric to a Riemannian product $S^k \times N$, where $S^k$ is a round sphere and $k > p$.

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As a further generalization, a $p$-form $\omega$ on an $n$-dimensional Riemannian manifold $(M, g)$ is called *conformal Killing–Yano* (CKY for short) if it satisfies the following equation:

$$\nabla_X \omega = \frac{1}{p+1}\, \iota_X d\omega - \frac{1}{n-p+1}\, X^* \wedge d^*\omega, \tag{2}$$

for any vector field $X$ on $M$, where $\nabla$ is the Levi-Civita connection, $X^*$ is the 1-form dual to $X$ and $d^* = (-1)^{n(p+1)+1} * d *$ is the co-differential. Note that a CKY form $\omega$ is co-closed, that is $d^*\omega = 0$, if and only if it is Killing–Yano.

We remark that the space of CKY forms is conformally invariant. Indeed, if $\omega$ is a CKY $p$-form on $(M, g)$ and $\tilde{g} := e^{2f} g$ is a conformally equivalent metric, then the form $\tilde{\omega} := e^{(p+1)f}\omega$ is a CKY $p$-form on $(M, \tilde{g})$ (see [6]). Moreover, this space of CKY forms is invariant by the Hodge-star operator (see [14]).

It was proved in [14] that on a compact 7-manifold with holonomy $G_2$ any CKY $p$-form with $p \neq 3, 4$ is parallel. The description of conformal Killing–Yano $p$-forms on a compact Riemannian product was obtained in [10], proving that such a form is a sum of forms of the following types: parallel forms, pull-back of Killing–Yano forms on the factors, and their Hodge duals.

In this article we will deal with 2-forms which are (conformal) Killing–Yano. We observe that the 2-form $\omega$ associated to a nearly Kähler manifold satisfies the Killing–Yano equation (1), while the canonical 2-form of a Sasakian manifold satisfies (2). Our goal is to construct examples of Riemannian manifolds carrying these distinguished 2-forms.

This article is organized as follows. In Section 2 we recall some basic results, and in Section 3 we study Killing–Yano 2-forms. We prove that on the total space of certain Riemannian submersions with totally geodesic fibers, for any $t > 0$ there exists a Riemannian metric $g_t$ admitting Killing–Yano 2-forms, extending results by Nagy on nearly Kähler structures in this class of manifolds ([11]). We also show a method to build Killing–Yano 2-forms on Lie groups with left invariant metrics, starting with a Lie group equipped with such a tensor and a suitable representation of its Lie algebra. In Section 4 we consider invariant conformal Killing–Yano 2-forms on Lie groups with left invariant metrics. One first obstruction obtained is that those forms occur in odd dimensions, provided that they are not Killing–Yano (see Theorem 4.3). Furthermore, imposing certain restrictions on the codifferential of such a form, then the center of the group is 1-dimensional and the quotient of the group by its center inherits a non-degenerate Killing–Yano 2-form (see Theorem 4.6). Using this, we show that skew-symmetric non-degenerate parallel 2-forms give rise to CKY 2-forms on a higher dimensional Lie group by considering central extensions. This leads to the study of such forms in Section 5, where we give some obstructions for their existence (see Theorem 5.1). Finally, in Section 6 we focus on the existence of (conformal) Killing–Yano 2-forms in two special classes of Lie groups with left invariant metrics: (i) Lie groups with flat left invariant metric, and (ii) almost abelian Lie groups, that is, Lie groups such that the corresponding Lie algebra has a codimension 1 abelian ideal.

## 2. Preliminaries

Let $(M, g)$ be a Riemannian manifold, and $T : TM \to TM$ a skew-symmetric endomorphism of the tangent bundle $TM$ of $M$ with its associated 2-form $\omega$ given by $\omega(X, Y) = g(TX, Y)$ for all $X, Y$ vector fields on $M$.

We denote by $N_T$ the Nijenhuis tensor of $T$, defined by

$$N_T(X, Y) := [TX, TY] - T([X, TY] + [TX, Y]) + T^2[X, Y] \tag{3}$$

and by $\nabla$ the Levi-Civita connection associated to $(M, g)$. Since

$$(\nabla_X T)\, Y = \nabla_X (TY) - T(\nabla_X Y), \tag{4}$$