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Multisymplectic 3-forms on 7-dimensional manifolds

Tomáš Salač

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ABSTRACT

A 3-form $\omega \in \Lambda^3\mathbb{R}^{7*}$ is called multisymplectic if it satisfies some natural non-degeneracy requirement. It is well known that there are 8 orbits (or types) of multisymplectic 3-forms on $\mathbb{R}^7$ under the canonical action of $GL(7, \mathbb{R})$ and that two types are open. This leads to 8 types of global multisymplectic 3-forms on 7-dimensional manifolds without boundary. The existence of a global multisymplectic 3-form of a fixed type is a classical problem in differential topology which is equivalent to the existence of a certain G-structure. The open types are the most interesting cases as they are equivalent to a G_2 and $\tilde{G}_2$-structure, respectively. The existence of these two structures is a well known and solved problem. In this article is solved (under some convenient assumptions) the problem of the existence of multisymplectic 3-forms of the remaining types.

1. Introduction

Put $V := \mathbb{R}^7$. There are finitely many orbits of the canonical action of $GL(V)$ on Λ^3V^*. We will call the orbits also types. A linear isomorphism $\Phi : \mathbb{R}^7 \to W$ induces a map $\Phi^* : \Lambda^3W^* \to \Lambda^3\mathbb{R}^{7*}$. The type of $\Phi^*\omega$ does not depend on the choice of linear isomorphism and thus, we can define the type for any skew-symmetric 3-form on any 7-dimensional real vector space.

A 3-form $\omega \in \Lambda^3V^*$ is called *multisymplectic* if the insertion map

$$V \to \Lambda^2V^*, \ v \mapsto i_v\omega := \omega(v, -, -) \tag{1}$$

is injective. There are (see [8] and [15]) eight types of multisympletic 3-forms and two of these types are open.

Let Ω be a global 3-form on a 7-dimensional manifold N without boundary and $i \in \{1, \ldots, 8\}$. We call Ω a *multisympletic 3-form of algebraic type i* if for each $x \in N$: Ω_x is a multisymplectic 3-form of type i. The existence of such a 3-form is a classical problem in differential topology, if O_i is the stabilizer of a fixed multisymplectic 3-form $\omega_i \in \Lambda^3V^*$ of algebraic type i, then N admits a multisymplectic 3-form of algebraic

E-mail address: salac@karlin.mff.cuni.cz.

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type i if, and only if it has an O_i-structure. The groups O_i were studied in [2] where they were given as semi-direct products of some well known Lie groups.

By the Cartan–Iwasawa–Malcev theorem (see [1, Theorem 1.2]), a connected Lie group H has a maximal compact subgroup and any two such subgroups are conjugated. Let us fix one such subgroup and let us denote it by K. By Cartan's result, the group H has the homotopy type of K and by a standard argument from the obstruction theory, any H-principal bundle reduces to a K-principal bundle. Hence, the first goal is (see Section 3) to find a maximal compact subgroup K_i of each group O_i. Then we solve (see Section 4) the problem of the existence of a multisympletic 3-form of algebraic type i on a given closed 7-manifold. The problem is not solved completely as for some types we assume that the underlying manifold is orientable or simply-connected.

The most interesting and well known cases are the types 8 and 5 as $O_8 = G_2$ and $O_5 = \tilde{G}_2$. The existence of a G_2-structure was solved in [10] and the existence of a $\tilde{G}_2$-structure in [11].

Let us summarize the main result of this article into a single Theorem. See Section 4.1 for the definition of characteristic classes $q(N)$ and $q(N;\ell)$.

Theorem 1.1. *Let N be a closed and connected 7-manifold.*

(1) *Suppose that N is orientable, spin^c and that there are $e, f \in H^2(N,\mathbb{Z})$ such that*

$$w_2(N) = \rho_2(e+f) \quad \text{and} \quad q(N; e+f) = -ef,$$

then N admits a multisymplectic 3-form of algebraic type 1.
If N is simply-connected, then the assumptions are also necessary.

(2) *Suppose that N is orientable, spin and that there are $e, f \in H^2(N,\mathbb{Z})$ such that*

$$-q(N) = e^2 + f^2 + 3ef,$$

then N admits a multisymplectic 3-form of algebraic type 2.
If N is simply-connected, then the assumptions are also necessary.

(3) *N admits a multisympletic 3-form of algebraic type* 3 *if, and only if N is orientable and spin^c.*

(4) *Suppose that N is orientable, spin and that there is $u \in H^4(N,\mathbb{Z})$ such that*

$$q(M) = -4u,$$

then N admits a multisymplectic 3-form of algebraic type 4.
On the other hand, if N admits a multisymplectic 3-form of algebraic type 4*, then N is orientable and spin.*

(5) *N admits a multisympletic 3-form of algebraic type $i = 5, 6, 7, 8$ if, and only if N is orientable and spin.*

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I would like to dedicate this article to M. Doubek who drew author's attention to this subject and who passed away in a car accident at the age of 33.

1.1. Notation

We will use the following notation: