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## Multisymplectic 3-forms on 7-dimensional manifolds

ABSTRACT

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To the memory of Martin Doubek

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### 1. Introduction

Put  $V := \mathbb{R}^7$ . There are finitely many orbits of the canonical action of GL(V) on  $\Lambda^3 V^*$ . We will call the orbits also types. A linear isomorphism  $\Phi : \mathbb{R}^7 \to W$  induces a map  $\Phi^* : \Lambda^3 W^* \to \Lambda^3 \mathbb{R}^{7*}$ . The type of  $\Phi^* \omega$  does not depend on the choice of linear isomorphism and thus, we can define the type for any skew-symmetric 3-form on any 7-dimensional real vector space.

A 3-form  $\omega \in \Lambda^3 \mathbf{V}^*$  is called *multisymplectic* if the insertion map

$$V \to \Lambda^2 V^*, \ v \mapsto i_v \omega := \omega(v, -, -)$$
 (1)

A 3-form  $\omega \in \Lambda^3 \mathbb{R}^{7*}$  is called multisymplectic if it satisfies some natural non-

degeneracy requirement. It is well known that there are 8 orbits (or types)

of multisymplectic 3-forms on  $\mathbb{R}^7$  under the canonical action of  $\mathrm{GL}(7,\mathbb{R})$  and

that two types are open. This leads to 8 types of global multisymplectic

3-forms on 7-dimensional manifolds without boundary. The existence of a global

multisymplectic 3-form of a fixed type is a classical problem in differential topology

which is equivalent to the existence of a certain G-structure. The open types are the most interesting cases as they are equivalent to a  $G_2$  and  $\tilde{G}_2$ -structure, respectively.

The existence of these two structures is a well known and solved problem. In this

article is solved (under some convenient assumptions) the problem of the existence

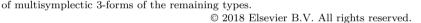
is injective. There are (see [8] and [15]) eight types of multisympletic 3-forms and two of these types are open.

Let  $\Omega$  be a global 3-form on a 7-dimensional manifold N without boundary and  $i \in \{1, \ldots, 8\}$ . We call  $\Omega$  a multisympletic 3-form of algebraic type i if for each  $x \in N$ :  $\Omega_x$  is a multisymplectic 3-form of type i. The existence of such a 3-form is a classical problem in differential topology, if  $O_i$  is the stabilizer of a fixed multisymplectic 3-form  $\omega_i \in \Lambda^3 V^*$  of algebraic type i, then N admits a multisymplectic 3-form of algebraic









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type *i* if, and only if it has an  $O_i$ -structure. The groups  $O_i$  were studied in [2] where they were given as semi-direct products of some well known Lie groups.

By the Cartan–Iwasawa–Malcev theorem (see [1, Theorem 1.2]), a connected Lie group H has a maximal compact subgroup and any two such subgroups are conjugated. Let us fix one such subgroup and let us denote it by K. By Cartan's result, the group H has the homotopy type of K and by a standard argument from the obstruction theory, any H-principal bundle reduces to a K-principal bundle. Hence, the first goal is (see Section 3) to find a maximal compact subgroup  $K_i$  of each group  $O_i$ . Then we solve (see Section 4) the problem of the existence of a multisympletic 3-form of algebraic type *i* on a given closed 7-manifold. The problem is not solved completely as for some types we assume that the underlying manifold is orientable or simply-connected.

The most interesting and well known cases are the types 8 and 5 as  $O_8 = G_2$  and  $O_5 = \tilde{G}_2$ . The existence of a  $G_2$ -structure was solved in [10] and the existence of a  $\tilde{G}_2$ -structure in [11].

Let us summarize the main result of this article into a single Theorem. See Section 4.1 for the definition of characteristic classes q(N) and  $q(N; \ell)$ .

**Theorem 1.1.** Let N be a closed and connected 7-manifold.

(1) Suppose that N is orientable, spin<sup>c</sup> and that there are  $e, f \in H^2(N, \mathbb{Z})$  such that

 $w_2(N) = \rho_2(e+f)$  and q(N; e+f) = -ef,

then N admits a multisymplectic 3-form of algebraic type 1.

If N is simply-connected, then the assumptions are also necessary.

(2) Suppose that N is orientable, spin and that there are  $e, f \in H^2(N, \mathbb{Z})$  such that

$$-q(N) = e^2 + f^2 + 3ef$$

then N admits a multisymplectic 3-form of algebraic type 2.

If N is simply-connected, then the assumptions are also necessary.

- (3) N admits a multisympletic 3-form of algebraic type 3 if, and only if N is orientable and spin<sup>c</sup>.
- (4) Suppose that N is orientable, spin and that there is  $u \in H^4(N, \mathbb{Z})$  such that

$$q(M) = -4u,$$

then N admits a multisymplectic 3-form of algebraic type 4.

On the other hand, if N admits a multisymplectic 3-form of algebraic type 4, then N is orientable and spin.

(5) N admits a multisympletic 3-form of algebraic type i = 5, 6, 7, 8 if, and only if N is orientable and spin.

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I would like to dedicate this article to M. Doubek who drew author's attention to this subject and who passed away in a car accident at the age of 33.

#### 1.1. Notation

We will use the following notation:

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