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Generalized Weingarten surfaces of harmonic type in hyperbolic 3-space

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In this paper we study a large class of Weingarten surfaces M with prescribed hyperbolic Gauss map in the Hyperbolic 3-space, which are the analogous to the Laguerre minimal surfaces in Euclidean space, these surfaces will be called Generalized Weingarten surfaces of harmonic type (HGW-surfaces), this class includes the surfaces of mean curvature one and the linear Weingarten surfaces of Bryant type (BLW-surfaces). We obtain a Weierstrass type representation for this surfaces which depend of three holomorphic functions. As applications we classify the HGW-surfaces of rotation and we obtain a Weierstrass type representation for surfaces of mean curvature one with prescribed hyperbolic Gauss map which depend of two holomorphic functions. Moreover, we classify a class of complete mean curvature one surfaces parametrized by lines of curvature whose coordinates curves has the same geodesic curvature up to sign.

1. Introduction

Bianchi in [2], study the surfaces of mean curvature one in hyperbolic space $\mathbb{H}^3$. During the last three decades these surfaces have called the attention of several authors (see for instance [5], [14], [16], [17], [19], [25–27], [29–31], [33] and [34]). Bryant in [4], gave a holomorphic parametrization of such surfaces, which is a close analogue of the Weierstrass representation for minimal surfaces in $\mathbb{R}^3$. Epstein in [14], study a class of surfaces whose mean curvature H and Gaussian curvature K_I satisfy a relation of the form $2\alpha(H-1)+(1-\alpha)K_I=0$, where $\alpha<0$.

More recently, Gálvez, Martinez and Milan in [17] extend Bryant's results and Epstein for surfaces whose mean curvature H and Gaussian curvature K_I satisfy a linear relation of the form $2a(H-1)+bK_I=0$, where $a+b\neq 0$, these surfaces are called *Linear Weingarten surfaces of Bryant type* (in short BLW-surfaces),

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the authors showed that given a BLW-surface, $\sigma = aI + bII$ is a Riemannian metric which induces a natural complex structure such that its hyperbolic Gauss map is a conformal map, we remark that when $\alpha = \frac{a}{a+b}$ the surfaces studied by Epstein are BLW-surfaces. When $a + b = 0$, that is, the non-elliptic case, are the ones with a constant principal curvature equal to 1. They are studied in [1]. Ferreira and Roitman in [15], studied hypersurfaces in the hyperbolic space whose r-curvatures satisfy a linear relation of Weingarten, which are related to equation of constant scalar curvature, in the case of surfaces they are BLW-surfaces.

Corro, in [5], introduce a large class of generalized Weingarten surfaces of Bryant type (in short, BGW-surfaces) in the hyperbolic 3-space, whose mean curvature H, Gaussian curvature K_I and radius function h satisfy a relation of the form

$$2ach^{2\frac{c-1}{c}}(H-1) + (a + b - ach^{2\frac{c-1}{c}})K_I = 0,$$

where $a + b \neq 0$ and $c \neq 0$, this class includes the BLW-surfaces ($c = 1$). Moreover, present a method for obtaining such surfaces in terms of holomorphic data, which is based on seeing a surface in hyperbolic 3-space, as the envelope of a sphere congruence whose other envelope is contained in the ideal boundary.

Schief [28], studied two classes of surfaces $S \subset \mathbb{R}^3$ satisfying a Weingarten relation of the form $(\mu^2 \pm \rho^2)K + 2\mu H + 1 = 0$, where $\mu, \rho : S \to \mathbb{R}^3$ are harmonic functions with respect to the quadratic form $\sigma = II + \mu III$, II and III are the second and third fundamental form of S, respectively.

In this paper we show that in all surface M in hyperbolic space $\mathbb{H}^3$ with principal curvatures different of one, the quadratic form $\sigma = -K_I I + 2(H-1)II$ defined a Riemannian metric, where I, II are the first and the second fundamental form of M, respectively.

Motivated by the papers [5] and [28], we introduce the *Generalized Weingarten surfaces of harmonic type* (in short HGW-surfaces), as those classes of surfaces M in hyperbolic space with hyperbolic Gauss map given by an immersion, with principal curvatures different of one and whose mean curvature H and Gaussian curvature K_I satisfies the relation

$$2Ce^{2\mu}(H-1) + (1 - Ce^{2\mu})K_I = 0,$$

where μ is a harmonic function with respect to the quadratic form $\sigma = -K_I I + 2(H-1)II$, $C \in \mathbb{R}$, $C \neq 0$. We present a method for obtaining such surfaces which depend on three holomorphic functions. Observe that when μ is constant we obtain BLW-surfaces, in particular, when $\mu = 0$ and $C = 1$ we obtain surfaces of mean curvature one and in these cases the method for constructing the surfaces turns out to be different from the one used in [4], [16], [17], [25], [29], [32] and [33].

On the other hand, if $X : M \to \mathbb{R}^3$ be an oriented surface with non-zero Gauss curvature K and mean curvature H. The functional $L(x) = \int_M (H^2 - K)/K dM$ is invariant under the Laguerre group. The critical surfaces of L are called Laguerre minimal surfaces (see Ref. [22] and Ref. [24]). The Euler–Lagrange equation of Laguerre minimal surfaces is given by Weingarten in 1888 as follows:

$$\triangle_{III}(R_1 + R_2) = 0,$$

where $R_i = \dfrac{1}{k_i}, i = 1, 2$ are curvature radii of X and $\triangle_{III}$ is the Laplacian operator with respect to the third fundamental form III of X. Therefore, the HGW-surfaces can be seen from a point of view similar to the case of Laguerre minimal surfaces, i.e. the HGW-surfaces satisfy the relation

$$\triangle_\sigma \ln|\widetilde{R}_1 + \widetilde{R}_2 - 1| = 0,$$

where $\widetilde{R}_i = \dfrac{1}{1 - k_i}, i = 1, 2$ are hyperbolic curvature radii of M defined in [13] and $\triangle_\sigma$ is the Laplacian operator with respect to the quadratic form $\sigma = -K_I I + 2(H-1)II$.