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Monochromatic metrics are generalized Berwald

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ABSTRACT

We show that monochromatic Finsler metrics, i.e., Finsler metrics such that each two tangent spaces are isomorphic as normed spaces, are generalized Berwald metrics, i.e., there exists an affine connection, possibly with torsion, that preserves the Finsler function.

1. Introduction

1.1. Definitions and main result

A *Finsler metric* on a smooth manifold M of dimension $n \geq 2$ is a function $F\colon TM \to [0, \infty)$ such that for every point $x \in M$ the restriction $F_x = F|_{T_xM}$ is a Minkowski norm. That means that

1. $F_x(\lambda\xi) = \lambda F_x(\xi)$ for all $\lambda \geq 0, \xi \in T_xM$
2. $F_x(\xi + \eta) \leq F_x(\xi) + F_x(\eta)$ for all $\xi, \eta \in T_xM$
3. $F_x(\xi) = 0 \Rightarrow \xi = 0$

We allow irreversibility and do not require strict convexity. We assume that F is smooth on the slit tangent bundle $TM \setminus \{0\}$.

Definition 1.1. A Finsler metric F on a connected manifold M is said to be a *generalized Berwald metric* if there exists a smooth affine connection ∇ on M, called *associated connection*, whose parallel transport preserves the Finsler function F.

That is, for every $x, y \in M$ and for any curve $\gamma\colon [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$ the parallel transport $P_\gamma^\nabla\colon T_xM \to T_yM$ along this curve is an isomorphism of the normed spaces (T_xM, F_x) and

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(T_yM, F_y) in the sense

$$F_y(P_\gamma^\nabla(\xi)) = F_x(\xi) \text{ for all } \xi \in T_xM.$$

In the definition above, the connection may have a torsion.

Definition 1.2. A Finsler metric F is called *monochromatic* if for every two points $x, y \in M$ there exists a linear isomorphism between the tangent spaces at these points which is an isometry with respect to F_x and F_y.

Clearly, as one of these points one can take some fixed point x_0, so monochromacy of a Finsler metric is equivalent to the existence of a field of linear isomorphisms $A_x := A(x)\colon T_{x_0}M \to T_xM$ such that

$$F(x, A_x(\xi)) = F(x_0, \xi) \text{ for all } \xi \in T_{x_0}M.$$

We do not assume that A_x depends smoothly or even continuously on x. This definition is due to David Bao [3] and is motivated by a suggestion of Zhongmin Shen, who proposed to assign a unique color to each Minkowski norm. Generic Finsler spaces are then "multicolored" because different points of the manifold will generically correspond to different colors. Monochromatic manifolds are such that all points correspond to the same color. Our main result is:

Theorem 1. *Let (M, F) be a connected Finsler manifold. Then, F is a generalized Berwald metric if and only if F is monochromatic.*

In Theorem 1 we assume that M is at least of class C^k and F is at least of class C^{k-1}, $k \geq 2$. From the proof it will be clear that the associated connection will be at least of class C^{k-2}. At the end of the paper, in §3, we discuss the existence of non-Riemannian generalized Berwald metrics on closed manifolds of small dimensions.

1.2. History and motivation

Definitions obviously similar to the definition of monochromatic metrics appeared many times in the literature, possibly first time in a commentary of Hermann Weyl on Riemann's habilitation address [12]. There, Weyl suggested to consider Finsler manifolds such that all tangent spaces are isomorphic as normed spaces. It is not clear though whether he assumed that the field of isomorphisms A_x in Definition 1.2 depends smoothly on x.

In 1965 Detlef Laugwitz [8] referred to Weyl's idea and suggested the following definition: he called a Finsler metric *metrically homogeneous* if, in our terminology, it is monochromatic and if in addition the field of linear isomorphisms A_x in our definition of a monochromatic metric depends smoothly on x. An equivalent definition was given by Yoshimiro Ichijyo [6] who called such Finsler metrics *Finsler metrics modeled on a Minkowski space.* Other equivalent definitions exist in the literature; such Finsler metrics were called *1-form metrics* in e.g. [9] and *affine deformations of Minkowski spaces* in e.g. [13].

It is easy to show and was independently done in [8, Exercise 15.4.1], [6, Theorem 2] and, quite recently, in [13, Theorem 1], that (locally) such metrics are generalized Berwald metrics. We essentially repeat their proof at the end of the proof of Theorem 1.

Many special cases of Theorem 1 were proved before. For example, [2] proved it for left-invariant Finsler metrics and [14] proves it for (α, β)-metrics such that the α-norm of β is constant (in this paper it is also assumed that the metric should satisfy the so called sign property, but actually by Theorem 1 this additional assumption can be omitted), see also [15].