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Almost complex structures on spheres

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ABSTRACT

In this paper we review the well-known fact that the only spheres admitting an almost complex structure are S^2 and S^6. The proof described here uses characteristic classes and the Bott periodicity theorem in topological K-theory. This paper originates from the talk "Almost Complex Structures on Spheres" given by the second author at the MAM1 workshop "(Non)-existence of complex structures on S^6", held in Marburg from March 27th to March 30th, 2017. It is a review paper, and as such no result is intended to be original. We tried to produce a clear, motivated and as much as possible self-contained exposition.

1. Introduction

A *complex manifold* is a differentiable manifold M whose transition functions are holomorphic. If $x + iy$ denotes local coordinates around $p \in M$, the multiplication by i induces an endomorphism J_p of the tangent space $T_p(M)$ given by

$$\begin{cases} \frac{\partial}{\partial_x}|_p \longmapsto \frac{\partial}{\partial_y}|_p \\ \frac{\partial}{\partial_y}|_p \longmapsto -\frac{\partial}{\partial_x}|_p \end{cases} \tag{1.1}$$

Writing the Cauchy–Riemann equations on the intersection of two overlapping charts, one obtains that J_p is globally defined. The endomorphism $J : T(M) \to T(M)$, locally defined by (1.1), is an example of almost

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complex structure. More generally, an *almost complex structure* on a real differentiable manifold M is a linear bundle morphism J of the tangent vector bundle $T(M)$ satisfying $J^2 = -\operatorname{Id}$. The pair (M, J) is called an *almost complex manifold*, and whenever J arises from (1.1) for certain holomorphic coordinates $x + iy$, one says that the almost complex structure J is *integrable*.

One often refers to the above discussion by the sentence: "on a complex manifold the multiplication by i gives a compatible integrable almost complex structure".

The problem of deciding if a given almost complex structure J is integrable is nowadays an easy task, due to the celebrated Newlander–Nirenberg Theorem [23]. Much more challenging is the problem of deciding whether a differentiable manifold admits *any* integrable almost complex structure: in real dimension > 4, the only known general obstruction is the trivial one, that is, the existence of an almost complex structure.

The existence of an almost complex structure J on a manifold M is possible only if the dimension m of M is even, because of $(\det J)^2 = (-1)^m$. Moreover, J is equivalent to a $\mathrm{GL}(m/2, \mathbb{C})$-structure on $T(M)$. In fact, any endomorphism J_p on the tangent space $T_p(M)$ such that $J_p^2 = -\operatorname{Id}$ admits an *adapted* basis $\{v, J(v)\}$, and with respect to this basis $J_p = \begin{pmatrix} 0, -1 \\ +1, 0 \end{pmatrix}$. Thus, all the J_p stick together giving an almost complex structure J if and only if the structure group $\mathrm{GL}(2n, \mathbb{R})$ can be reduced to $\mathrm{GL}(n, \mathbb{C})$. See [7, Remark 2 at page 8] for a nice discussion regarding the terminology "almost" and "integrable" for G-structures.

Very clear references for basic facts on almost complex structures are the classical Kobayashi–Nomizu [17, IX.1 and IX.2] and the Besse [3, 2.A].

This paper originates from the talk "Almost Complex Structures on Spheres" given by the second author at the MAM1 workshop "(Non)-existence of complex structures on S^6", held in Marburg from March 27th to March 30th, 2017. The plan of this beautiful workshop was to discuss and summarize various results of the past 30 years about the existence or non-existence of integrable almost complex structures on spheres, and this talk was about the classical result of Borel and Serre ruling out all spheres besides S^2 and S^6, namely, the result from where the story began:

Theorem 1.1 *(Borel and Serre, 1953 [8]). The sphere S^n admits an almost complex structure if and only if $n = 2$ or $n = 6$.*

The material in this paper comes from several sources, clearly stated whenever necessary. As such, no result is intended to be original. However, we tried to produce a clear, motivated and as much as possible self-contained exposition.

2. Almost complex structures on S^2 and S^6

An almost complex structure on S^2 can be constructed in several ways. One can observe that since S^2 is orientable, its structure group $\mathrm{GL}(2, \mathbb{R})$ can be reduced to $\mathrm{SO}(2) = \mathrm{U}(1) \subset \mathrm{GL}(1, \mathbb{C})$. Or, one can observe that S^2 is diffeomorphic to $\mathbb{C}P^1$.

An explicit construction of J is possible by means of a vector product in R^3. The feasibility of this approach, that we are now going to describe, is that the same construction leads to an almost complex structure on S^6.

Denote by $\mathbb{H}$ and $\mathbb{O}$ the quaternions and the octonions, respectively. We will denote the product of $u, v \in \mathbb{H}$ or $u, v \in \mathbb{O}$ by juxtaposition uv, and the conjugation by u^*. Note then that $(uv)^* = v^* u^*$, and $uu^* = |u|^2$, where $|\cdot|$ is the standard Euclidean norm. By polarization, we get $2\langle u, v\rangle = uv^* + v^* u$, which implies the useful identity $uv^* = -v^* u$ whenever $u \perp v$. On the octonions, we have a non-trivial *associator*, that is, $[u, v, w] \stackrel{\text{def}}{=} ((uv)w) - (u(vw))$ is not identically zero if $u, v, w \in \mathbb{O}$. A key property of this associator is that it is *alternating*, that is, $[u, v, w] = 0$ whenever two amongst u, v, w are equal.

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