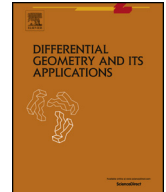




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Non-existence of orthogonal complex structures on the round 6-sphere

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ABSTRACT

In this short note, we review the well-known result that there is no orthogonal complex structure on S^6 with respect to the round metric.

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1. Introduction

The volume where this note appears is dedicated to the famous Hopf problem, that is, the question whether there is a complex structure on the 6-sphere.

Here, we will focus on the round 6-sphere, i.e. S^6 equipped with its notable metric of constant sectional curvature (unique up to rescaling). This metric is inherited from the ambient Euclidean metric on \mathbb{R}^7 and, under the stereographic projection $S^6 \setminus \{\infty\} \rightarrow \mathbb{R}^6$, can be written in coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$ as

$$g = 4 \frac{dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 + dx_6^2}{(1 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)^2}$$

(with this scaling factor, its sectional curvature is constant and equal to 1).

Recall that an almost complex structure J on S^6 is an endomorphism of its tangent bundle $J : TS^6 \rightarrow TS^6$ such that its square is minus the identity on each fiber, that is, $J_x^2 = -\text{Id}_x$, for all $x \in S^6$. The almost

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complex structure is said to be orthogonal (with respect to the round metric) if it acts as an isometry of $T\mathbb{S}^6$, more concretely, if $g(JX, JY) = g(X, Y)$, for all vector fields of $T\mathbb{S}^6$. The almost complex structure J is said to be integrable if \mathbb{S}^6 becomes a complex manifold such that local charts can be found in which J corresponds to multiplication by $i = \sqrt{-1}$ in \mathbb{C}^3 . In this case, we say that J is a complex structure and that (g, J) is a Hermitian structure on \mathbb{S}^6 .

The main objective of this article is to review the proofs, placing them in their historical context, of the following result.

Theorem 1. *There is no orthogonal almost complex structure on the round 6-sphere which is integrable.*

As far as the author can establish, it was widely believed in the mathematical community that this result had been first proven by Claude LeBrun in 1987, [5]. The author became aware that this was not the case through a comment of Robert Bryant on *Math Overflow*, [6]. The same result had already been established more than thirty years earlier by André Blanchard using ideas which anticipated twistors, [3].

2. The article of LeBrun

The article [5] consists of a three-page text that presents a concise and ingenious proof of Theorem 1. On the one hand, the proof is elementary in that it does not involve any hard “machinery” but, on the other hand, since the arguments are very specific to the case under consideration it cannot be trivially generalized to other situations. In this section, we will work out this proof backwards and fill in some of the details.

The proof is set by contradiction. Suppose that there exists an orthogonal complex structure on \mathbb{S}^6 . The strategy is to exhibit an embedding $\tau : \mathbb{S}^6 \hookrightarrow M$ where M is a Kähler manifold and τ is a holomorphic map.

Recall that the De Rham cohomology of spheres is as follows:

$$H_{DR}^k(\mathbb{S}^m) = \begin{cases} \mathbb{R}, & \text{if } k = 0, m \\ 0, & \text{otherwise.} \end{cases}$$

By exhibiting the map τ , \mathbb{S}^6 would become a complex submanifold of a Kähler manifold and, therefore, would also be a Kähler manifold. However, for such manifolds the Hermitian form $\Omega(X, Y) = g(JX, Y)$ is closed and its cohomology class $[\Omega]$ is a generator of degree 2. But, since $H_{DR}^2(\mathbb{S}^6) = 0$, this cannot happen.

The manifold M considered is $\text{Gr}_3(\mathbb{C}^7)$, the complex Grassmannian of 3-planes in \mathbb{C}^7 . Grassmannians (real or complex) are well-studied examples of manifolds. For $k, n \in \mathbb{N}$, $\text{Gr}_k(\mathbb{C}^n)$ is the set of all complex k -dimensional linear subspaces of \mathbb{C}^n . For instance, if $k = 1$, then $\text{Gr}_1(\mathbb{C}^n) = \mathbb{C}P^{n-1}$. Fixing a Hermitian product on \mathbb{C}^n , we can write the Grassmannians as classical homogeneous spaces:

$$\text{Gr}_k(\mathbb{C}^n) = \frac{U(n)}{U(k) \times U(n-k)}.$$

We can readily check the duality $\text{Gr}_k(\mathbb{C}^n) = \text{Gr}_{n-k}(\mathbb{C}^n)$ and that their complex dimension is equal to $k(n-k)$. Complex Grassmannians are Kähler manifolds and, as with many properties of such manifolds, the tangent spaces are canonical/tautological objects. More precisely, at each k -plane K of \mathbb{C}^n its tangent space is $T_K \text{Gr}_k(\mathbb{C}^n) \simeq \text{Hom}(K, \mathbb{C}^n/K)$, the vector space of linear maps from K to \mathbb{C}^n/K .

Let J be any almost complex structure on \mathbb{S}^6 and fix $x \in \mathbb{S}^6$. For simplicity of notation, we will write T instead of $T\mathbb{S}^6$ and T_x for the fiber at x . Since $J_x : T_x \rightarrow T_x$ is such that $J_x^2 = -\text{Id}_x$, the eigenvalues of its complexification $J : T \otimes \mathbb{C} \rightarrow T \otimes \mathbb{C}$ are $+i$ and $-i$. We can define 3-dimensional vector subbundles as follows:

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