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Non-existence of orthogonal complex structures on $\mathbb{S}^6$ with a metric close to the round one

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ABSTRACT

I review several proofs for non-existence of orthogonal complex structures on the six-sphere, most notably by G. Bor and L. Hernández-Lamoneda, but also by K. Sekigawa and L. Vanhecke that we generalize for metrics close to the round one. Invited talk at MAM-1 workshop, 27–30 March 2017, Marburg.

1. Introduction

In 1987 LeBrun[1] [7] proved the following restricted non-existence result for the 6-sphere. Let (M, g) be a connected oriented Riemannian manifold. Denote by $\mathcal{J}_g(M)$ the space of almost complex structures J on M that are compatible with the metric (i.e. $J^*g = g$) and with the orientation. This is the space of sections of an $SO(6)/U(3)$ fiber bundle, so whenever non-empty it is infinite-dimensional. Associating to $J \in \mathcal{J}_g(M)$ the almost symplectic structure $\omega(X, Y) = g(JX, Y)$, $X, Y \in TM$, we get a bijection between $\mathcal{J}_g(M)$ and the space of almost Hermitian triples (g, J, ω) on M with fixed g.

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[1] As discussed in the paper by A.C. Ferreira in this volume, the same result was established in 1953 by A. Blanchard [1] using the ideas anticipating twistors. The proofs of [1,7] are reviewed there.

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Theorem 1. *No $J \in \mathcal{J}_{g_0}(\mathbb{S}^6)$ is integrable (is a complex structure) for the standard (round) metric g_0. In other words, there are no Hermitian structures on $\mathbb{S}^6$ associated to the metric g_0.*

There are several proofs of this statement, we are going to review some of those. The method of proof of Theorem 1 by Salamon [8] uses the fact that the twistor space of $(\mathbb{S}^6, g_0)$ is $\mathcal{Z}(\mathbb{S}^6) = SO(8)/U(4)$ which is a Kähler manifold (it has a complex structure because $\mathbb{S}^6$ is conformally flat, and the metric is induced by g_0), and so the holomorphic embedding $s_J : \mathbb{S}^6 \to \mathcal{Z}(\mathbb{S}^6)$ would induce a Kähler structure on $\mathbb{S}^6$.

Here the symmetry of g_0 is used (homogeneity), so this proof is not applicable for $g \approx g_0$ (but as mentioned in [2], a modification of the original approach of [7], based on an isometric embedding of $(\mathbb{S}^6, g)$ into a higher-dimensional Euclidean space, is possible).

A generalization of Theorem 1 obtained in [2] is as follows.

Theorem 2. *Let g be a Riemannian metric on $\mathbb{S}^6$. Denote by R_g its Riemannian curvature, considered as a $(3,1)$-tensor, and by $\tilde{R}_g : \Lambda^2 T^*\mathbb{S}^6 \to \Lambda^2 T^*\mathbb{S}^6$ the associated $(2,2)$ tensor (curvature operator). Assume that its spectrum (15 functions λ_i on $\mathbb{S}^6$ counted with multiplicities) $\mathrm{Sp}(\tilde{R}_g) = \{\lambda_{\min} \le \dots \le \lambda_{\max}\}$ is positive $\lambda_{\min} > 0$ and satisfies $5\lambda_{\max} < 7\lambda_{\min}$. Then no $J \in \mathcal{J}_g(\mathbb{S}^6)$ is integrable.*

This theorem will be proven in Section 4 after we introduce the notations and recall the required knowledge in Sections 2 and 3. Then we will give another proof of Theorem 1 due to Sekigawa and Vanhecke [9] in Section 5 (there is a related approach in [11]). Then in Section 6 we generalize it in the spirit of Theorem 2. Section 7 will be a short summary and an outlook.

Let us start with an alternative (to [7]) proof of Theorem 1 following Bor and Hernández-Lamoneda [2].

Sketch of the proof of Theorem 1. Let $K = \Lambda^{3,0}(\mathbb{S}^6)$ be the canonical line bundle of the hypothetical complex structure J. Equip it with the Levi-Civita connection ∇ that is induced from $\Lambda^3_{\mathbb{C}}(\mathbb{S}^6)$ by the orthogonal projection. The curvature of K with respect to ∇ is

$$\Omega = R_\nabla|_{\Lambda^{3,0}} + \Phi^* \wedge \Phi = i\tilde{R}_g(\omega) + \Phi^* \wedge \Phi, \tag{1}$$

where Φ is the second fundamental form (see §3.1). It has type $(1,0)$ and so $i\Phi^* \wedge \Phi \le 0$ (see §3.2). Since for the round metric $g = g_0$ we have $\tilde{R}_g = \mathrm{Id}$, so

$$i\Omega = -\omega + i\Phi^* \wedge \Phi < 0.$$

Thus $-i\Omega$ is a non-degenerate (positive) scalar valued 2-form which is closed by the Bianchi identity. This implies that $\mathbb{S}^6$ is symplectic which is impossible due to $H^2_{\mathrm{dR}}(\mathbb{S}^6) = 0$.

It is clear from the proof that for $g \approx g_0$ the operator $R_g \approx \mathrm{Id}$ is still positive, so the conclusion holds for a small ball around g_0 in $\Gamma(\odot^2_+ T^*\mathbb{S}^6)$. It only remains to justify the quantitative claim.

2. Background I: connections on Hermitian bundles

Let M be a complex n-dimensional manifold. In this section we collect the facts about calculus on M important for the proof. A hurried reader should proceed to the next section returning here for reference.

Let $\pi : E \to M$ be a Hermitian vector bundle, that is a holomorphic bundle over M equipped with the Riemannian structure $\langle , \rangle$ in fibers for which the complex structure J in the fibers is orthogonal. Examples are the tangent bundle TM and the canonical line bundle $K = \Lambda^{n,0}(M)$.

Note that a Hermitian structure is given via a $\mathbb{C}$-bilinear symmetric product $\odot^2(E \otimes \mathbb{C}) \to \mathbb{C}$ as follows: the restriction $(,) : E' \otimes E'' \to \mathbb{C}$, where $E \otimes \mathbb{C} = E' \oplus E'' = E_{(1,0)} \oplus E_{(0,1)}$ is the canonical decomposition into $+i$ and $-i$ eigenspaces of the operator J, gives the Hermitian metric $\langle , \rangle : E \otimes E \to \mathbb{C}$, $\langle \xi, \eta \rangle = (\xi, \bar{\eta})$.

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