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Exceptionally simple PDE

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ABSTRACT

We give local descriptions of parabolic contact structures and show how their flat models yield explicit PDE having symmetry algebras isomorphic to all complex simple Lie algebras except $\mathfrak{sl}_2$. This yields a remarkably uniform generalization of the Cartan–Engel models from 1893 in the G_2 case. We give a formula for the harmonic curvature of a G_2-contact structure and describe submaximally symmetric models for general G-contact structures.

1. Introduction

The Cartan–Killing classification of all complex simple Lie algebras was one of the great milestones of 19th century mathematics. In addition to the classical series of type $A_\ell, B_\ell, C_\ell, D_\ell$ (corresponding to the complex matrix Lie algebras $\mathfrak{sl}_{\ell+1}, \mathfrak{so}_{2\ell+1}, \mathfrak{sp}_{2\ell}, \mathfrak{so}_{2\ell}$), five surprising "exceptional" Lie algebras of type G_2, F_4, E_6, E_7, E_8 of dimensions 14, 52, 78, 133, 248 were discovered. Since Lie algebras arose from the study of transformation groups, one can naturally ask for geometric structures whose symmetry algebra is a given simple Lie algebra. In 1893, Cartan [5] and Engel [11] announced the first explicit (local) geometric realizations for G_2 (see Table 1), most of which can be formulated as differential equations.

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Table 1
The Cartan–Engel G_2 models.

Dim	Geometric structure	Model
7	Parabolic Goursat PDE $\mathcal{F}$	$9(u_{xx})^2 + 12(u_{yy})^2(u_{xx}u_{yy} - (u_{xy})^2) + 32(u_{xy})^3 - 36u_{xx}u_{xy}u_{yy} = 0$
6	Involutive pair of PDE $\mathcal{E}$	$u_{xx} = \frac{1}{3}(u_{yy})^3, \quad u_{xy} = \frac{1}{2}(u_{yy})^2$
5	$(2,3,5)$-distribution $\overline{\mathcal{E}}$	$dx_2 - x_4dx_1,\ dx_3 - x_2dx_1,\ dx_5 - x_4dx_2$ (equivalently, Hilbert–Cartan: $Z' = (U'')^2$)
5	G_2-contact structure (contact twisted cubic field)	$\begin{cases} dz + x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2 = 0, \\ dx_2^2 + \sqrt{3}dy_1dy_2 = 0, \\ dx_2dy_2 - 3dx_1dy_1 = 0, \\ dy_2^2 + \sqrt{3}dx_1dx_2 = 0 \end{cases}$

Later, in his 5-variables paper [7], Cartan established remarkable correspondences between:

- contact (external) symmetries of (non-Monge–Ampère) parabolic Goursat PDE in the plane;
- contact (external) symmetries of nonlinear involutive pairs of PDE in the plane;
- symmetries of $(2,3,5)$-distributions.

In a tour-de-force application of his method of equivalence, Cartan then solved the equivalence problem for $(2,3,5)$-distributions. Nowadays, we formalize this as a (regular, normal) parabolic geometry of type (G_2, P_1). (For the parabolic subgroup $P_1 \subset G_2$, see “Conventions” below.) This yields a notion of curvature for such geometries and there is a (locally) unique “flat” model with maximal symmetry dimension $\dim(G_2) = 14$. The 1893 G_2-models $\overline{\mathcal{E}}, \mathcal{E}, \mathcal{F}$ are associated to the flat case of this general curved story.

Yamaguchi [30] generalized the reduction theorems underlying Cartan's correspondences in [7,8]. For all $G \neq A_\ell, C_\ell$, he identified the reduced geometries analogous to G_2/P_1 (see [30, pg. 310]) and proved the existence of corresponding (nonlinear) PDE admitting external symmetry $\mathfrak{g}$. However, these PDE were not *explicitly* described.[1] Exhibiting these models is one of the results of our article.

Notably absent in the Cartan–Yamaguchi story is Engel's 1893 model, namely a contact 5-manifold whose contact distribution is endowed with a twisted cubic field, which is the flat model for *G_2-contact structures*, i.e. G_2/P_2 geometries. Our article will focus on its generalization to structures called *G-contact structures* (or *parabolic contact structures*), modelled on the adjoint variety $G/P \cong G^{ad} \hookrightarrow \mathbb{P}(\mathfrak{g})$ of a (connected) complex simple Lie group G. This adjoint variety is always a complex contact manifold except for $A_1/P_1 \cong \mathbb{P}^1$, so $G = A_1 \cong \mathrm{SL}_2$ will be henceforth excluded. Letting $\dim(G/P) = 2n+1$, a G-contact structure consists of a contact manifold $(M^{2n+1}, \mathcal{C})$ (locally, the first jet-space $J^1(\mathbb{C}^n, \mathbb{C})$) with $\mathcal{C}$ (a field of conformal symplectic spaces) equipped with additional geometric data.

Restrict now to $G \neq A_\ell, C_\ell$. Earlier formulations of G-contact structures identified $\mathcal{C}$ as a tensor product of one or more auxiliary vector bundles: in the G_2 case, $\mathcal{C} \cong S^3E$ where $E \to M$ has rank two, and similarly for the exceptional cases [4, §4.2.8]; for the B_ℓ, D_ℓ cases (Lie contact structures), see [25]. While these abstract descriptions were sufficient for solving the equivalence problem, no concrete local descriptions were given in these works. Recently, a local description in terms of a conformal quartic tensor $[\mathsf{Q}]$ on $\mathcal{C}$ was used by Nurowski [22] and Leistner et al. [18]. But this viewpoint does not naturally lead to PDE.

We start from Engel's algebro-geometric perspective: G-contact structures can be described in terms of a *sub-adjoint variety field* $\mathcal{V} \subset \mathbb{P}(\mathcal{C})$. But $\mathcal{V}$ naturally induces other fields $\widehat{\mathcal{V}} \subset \widetilde{\mathcal{V}} \subset M^{(1)}$ and $\tau(\mathcal{V}) = \{\mathsf{Q} = 0\} \subset \mathbb{P}(\mathcal{C})$, and it turns out that these essentially give equivalent descriptions of the same G-contact structure. In particular, their symmetry algebras are the *same*. Here, $M^{(1)} \to M$ is the *Lagrange–Grassmann bundle*, whose fibre over $m \in M$ is the Lagrangian–Grassmannian $\mathrm{LG}(\mathcal{C}_m)$. Locally, $M^{(1)}$ is isomorphic to the second

[1] In [31, Sec. 6.3], Yamaguchi gave explicit *linear* PDE with E_6 and E_7 symmetry, but these are not the PDE from [30].