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Isoperimetric inequality along the twisted Kähler–Ricci flow

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ABSTRACT

We prove a uniform isoperimetric inequality for all time along the twisted Kähler–Ricci flow on Fano manifolds.

1. Introduction

The classical *isoperimetric inequality* states that for Borel set $\Omega \in \mathbb{R}^n (n \geq 2)$ with finite Lebesgue measure $|\Omega|$, the ball with the same measure has a lower perimeter, that is,

$$P(\Omega) \geq n\omega_n^{\frac{1}{n}} |\Omega|^{\frac{n-1}{n}}, \tag{1.1}$$

where $P(\Omega)$ is the distributional perimeter of Ω which coincides with the classical $n-1$-dimensional area of $\partial\Omega$ if Ω has smooth boundary and ω_n is the volume of unit ball in $\mathbb{R}^n$. It is also well-known that equality holds in (1.1) if and only if Ω is a ball B in $\mathbb{R}^n$. De Giorgi [14] (see also [15] for English version) proved

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(1.1) for the first time in the general framework of sets with finite perimeter. One can find various kinds of proofs and different formulations of the isoperimetric inequality in [1,2,4,11,19,24] and references therein.

In the case of geometric flows, Hamilton [17] obtained an isoperimetric estimate for the Ricci flow on the two sphere. For complex 2-dimensional Kähler–Ricci flow, Chen and Wang [5] proved that the isoperimetric constant for $(M,\, g(t))$ is bounded from below by a uniform constant. Here $g(t)$ is the solution of the Kähler–Ricci flow (see (1.2) with $\theta_{i\bar{j}} \equiv 0$). Later, Tian and Zhang [25] proved that, for all complex n-dimensional Kähler–Ricci flow on Fano manifolds, the isoperimetric constant for $(M,\, g(t))$ is also bounded from below by a uniform constant.

In this paper, we obtain a uniform estimate of lower bound on isoperimetric constant along the twisted Kähler–Ricci flow on Fano manifolds. To be precise, we need some notations and definitions. Let M be a real $n(=2m)$ dimensional Fano manifold with Kähler form ω_0 associated to the Kähler metric g_0. We consider the twisted Kähler–Ricci flow (see [9,18,30] and the references therein)

$$\begin{cases} \frac{\partial}{\partial t} g_{i\bar{j}}(x,t) = -R_{i\bar{j}}(x,t) + \theta_{i\bar{j}}(x) + g_{i\bar{j}}(x,t), \\ \quad g_{i\bar{j}}(x,0) = (g_0)_{i\bar{j}}(x), \end{cases} \tag{1.2}$$

where θ is a closed semi-positive $(1,1)$ form and

$$[2\pi c_1(M)] = [\omega(x,t) + \theta].$$

Here $\omega(x,t) = \sqrt{-1} g_{i\bar{j}}(x,t) \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j$ associated to the Kähler metric $g(x,t)$. For convenience, we denote

$$\mathcal{S}_{i\bar{j}}(x,t) = R_{i\bar{j}}(x,t) - \theta_{i\bar{j}}(x)$$

and

$$S(x,\, t) = 2 \sum_{i,j=1}^{m} g^{\bar{j}i}(x,\, t) \mathcal{S}_{i\bar{j}}(x,\, t).$$

We know that

Proposition 1.1. *For the twisted Kähler–Ricci flow (1.2) on Fano manifolds, there exist uniform positive constants C, κ and C_S such that*

(a) $|S(x,t)| \le C$,
(b) $|\mathrm{diam}(M,\, g(t))| \le C$,
(c) $\|h\|_{C^1} \le C$, *where from $\partial\bar{\partial}$-lemma, $h \in C^\infty(M, \mathbb{R})$ satisfies*

$$\mathcal{S}_{i\bar{j}} - g_{i\bar{j}} = \partial_i \partial_{\bar{j}} h, \tag{1.3}$$

(d) $\mathrm{Vol}_{g(t)}(B(x,r,t)) \ge \kappa r^n$, *for any $t > 0$ and $r \in (0,\, \mathrm{diam}(M,\, g(t)))$,*
(e) $\mathrm{Vol}_{g(t)}(B(x,r,t)) \le \kappa^{-1} r^n$, *for any $t > 0$ and $r > 0$,*
(f) *for any $f \in W^{1,2}(M)$,*

$$\left(\int_M |f|^{\frac{2n}{n-2}} \mathrm{d}\mu(t) \right)^{\frac{n-2}{n}} \le C_S \left(\int_M \left[|\nabla f|^2_{g(t)} + f^2 \right] \mathrm{d}\mu(t) \right).$$