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The differential geometry of curves in the Heisenberg groups

Hung-Lin Chiu [a], XiuHong Feng [b], Yen-Chang Huang [c,*]

[a] *Department of Mathematics, National Central University, Taiwan, R.O.C.*
[b] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, China*
[c] *School of Mathematics and Statistics, Xinyang Normal University, China*

A R T I C L E   I N F O

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A B S T R A C T

We study the horizontally regular curves in the Heisenberg groups H_n. We prove a fundamental theorem for curves in H_n $(n \geq 1)$ and define the order of horizontally regular curves. We also show that the curve γ is of order k if and only if, up to a Heisenberg rigid motion, γ lies in H_k but not in H_{k-1}; moreover, two curves with the same order differ in a rigid motion if and only if they have the same invariants: p-curvatures and contact normality. Thus, combining these results with our previous work [3] we get a complete classification of horizontally regular curves in H_n for $n \geq 1$.

1. Introduction

The Heisenberg groups are the standard model in the following sense: just like the role Euclidean spaces play in Riemannian geometry, the Heisenberg groups are the flat model of pseudohermitian manifolds, which are differential manifolds equipped with a CR structure as well as a contact form. To completely approach pseudohermitian manifolds, we focus on the study of the model. For a complete overview of Heisenberg groups, we refer the reader to the book [2]. The problem in this paper arose from [6] where the authors showed that the p-mean curvature of a p-minimal graph is the line curvature of a characteristic curves (see also [7]). Such the local result reflects the significance of studying curves and areas for embedded surfaces in H_1, for instance, the study of the isoperimetric inequality in H_1 in [1,8,11].

Beyond the local results, one of the main goals in studying differential geometry is to understand the global behavior of manifolds. Actually, as we know, there always exists local foundations hidden in global results. Therefore, in order to systematically develop the global geometry, the local foundations must be well-studied in advance. The geometry of submanifolds in the Heisenberg groups plays the role of such

* Corresponding author.
E-mail addresses: hlchiu@math.ncu.edu.tw (H.-L. Chiu), fengxiuhong@nuist.edu.cn (X.-H Feng), ychuang@xynu.edu.cn (Y.-C. Huang).

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foundation. Our strategy in a series of papers is to obtain a complete set of (local) invariants for a variety of submanifolds, and to see how we can go from local to global results.

In Euclidean geometry, the local geometry is completely characterized by the elegant fundamental theorem, which states that there are only three invariants for submanifolds in Euclidean spaces: the first fundamental form (the intrinsic one), the second fundamental form and the normal connections [4,12]. Although the Heisenberg group H_n coincides, as a set, with $\mathbb{R}^{2n+1}$, the set of invariants becomes much more complicated. The main reason is that there is a smooth contact structure defining the sub-Riemannian structure of the Heisenberg group. Henceforth, when studying submanifolds in the Heisenberg groups, one must take into account how the submanifolds are placed into the Heisenberg groups, namely, the positions of the submanifolds with respect to the contact structure have to be carefully dealt with. We also have to consider what the intrinsic structure is on the submanifolds induced from the ambient spaces, the Heisenberg groups. In conclusion, the geometry of the Heisenberg groups is much more rich than the one in Euclidean spaces.

This is one in our series of papers addressing with the local geometry of the Heisenberg groups. In [5], we consider hypersurfaces, and give a complete system of invariants for the non-singular part. In this paper, we focus on curves in the Heisenberg groups. In addition, we introduce the concept of order for curves in the Heisenberg groups. Note that all methods in this paper work, without changes, for the submanifolds in Euclidean spaces. The studies of hypersurfaces with singular points, pseudohermitian submanifolds (i.e., submanifolds with the pseudohermitian structure as its induced intrinsic structure), and Legendrian submanifolds will appear in forthcoming papers.

Now we give a brief introduction to the Heisenberg groups. The Heisenberg group H_n, $n \geq 1$, is the space $\mathbb{R}^{2n+1}$ with the group multiplication

$$\begin{aligned}&(x_1, \cdots, x_n, y_1, \cdots, y_n, z) \circ (\widetilde{x}_1, \cdots, \widetilde{x}_n, \widetilde{y}_1, \cdots, \widetilde{y}_n, \widetilde{z})\\ =&(x_1 + \widetilde{x}_1, \cdots, x_n + \widetilde{x}_n, y_1 + \widetilde{y}_1, \cdots, y_n + \widetilde{y}_n, z + \widetilde{z} + \sum_{j=1}^{n}(y_j\widetilde{x}_j - x_j\widetilde{y}_j)).\end{aligned}$$

It is a $(2n+1)$-dimensional Lie group, and the space of all left invariant vector fields is spanned by the basic vector fields:

$$\mathring{e}_j = \frac{\partial}{\partial x_j} + y_j\frac{\partial}{\partial z}, \quad \mathring{e}_{n+j} = \frac{\partial}{\partial y_j} - x_j\frac{\partial}{\partial z}, \quad T = \frac{\partial}{\partial z},$$

for $1 \leq j \leq n$.

The Heisenberg group H_n can be regarded as a pseudohermitian manifold with zero Webster-curvature and torsion. For more details, the reader is referred to the Appendix in [6] or [3], [5], [7], [10]. We give a brief description of geometric structures on H_n: the standard contact bundle in H_n is the subbundle ξ of the tangent bundle TH_n spanned by $\mathring{e}_j$ and $\mathring{e}_{n+j}$ for $1 \leq j \leq n$. The contact bundle can also be defined as the kernel of the contact form

$$\theta = dz + \sum_{j=1}^{n}(x_j dy_j - y_j dx_j).$$

The standard CR structure on H_n is the almost complex structure defined on ξ by

$$J(\mathring{e}_j) = \mathring{e}_{n+j}, \; J\mathring{e}_{n+j} = -\mathring{e}_j, \text{ for } j = 1, \cdots, n.$$

Throughout the article, we regard the Heisenberg group H_n with the standard pseudo-hermitian structure (J, θ) as a pseudohermitian manifold (H_n, J, θ). Denote the group of pseudohermitian transformations on