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Explicit Solutions to the mean field equations on hyperelliptic curves of genus two

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ABSTRACT

Let X be a complex hyperelliptic curve of genus two equipped with the canonical metric ds^2. We study mean field equations on complex hyperelliptic curves and show that the Gaussian curvature function of (X, ds^2) determines an explicit solution to a mean field equation.

1. Introduction

Mean field equations came originally from the study of prescribing (Gaussian) curvature problems in differential geometry. In [2], [3], Lin and Wang studied the mean field equation of the following type

$$\Delta u + \rho e^u = \rho \delta_0, \quad \rho \in \mathbb{R}_+ \tag{1.1}$$

on a flat torus T where Δ is the Laplace–Beltrami operator on T. They discovered in [2] that when $\rho = 8\pi$, (1.1) has a solution if and only if the set of critical points of the Green's function on T contains points other than the three half-period points. In a recent paper [1], Chai and Lin and Wang showed that when

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$\rho = 4\pi(2n+1)$, where n is nonnegative integer, the number of solutions to (1.1) is $n+1$ except for a finite number of conformal isomorphism classes of flat tori and when $\rho = 8\pi n$, where n is a positive integer, the solvability of (1.1) depends on the moduli space of flat tori. In this article, we consider the following mean field equation

$$\Delta u + \rho e^u = 4\pi \sum_{i=1}^{m} n_i \delta_{P_i} \tag{1.2}$$

on a compact Riemann surface X of genus two with a hermitian metric ds^2, where ρ and n_i are some integers and $\{P_i\}$ are distinct points on X. When ds^2 is the canonical metric on X, we can prove that the Gaussian curvature function of (X, ds^2) determines an explicit solution to (1.2). In this case, $m = 6$ and $n_i = 1$ and P_i are the Weierstrass points of X for $1 \leq i \leq 6$. Furthermore, we also discover that given any finite subset of distinct points $\{Q_1, \cdots, Q_s\}$ of X and any finite set of natural numbers $\{m_1, \cdots, m_s\}$, for a certain choice of nonnegative continuous function h on X, there exists a real valued function v defined and smooth on $X \setminus \{Q_1, \cdots, Q_s, P_1, \cdots, P_6\}$ so that v satisfies the following mean field equation

$$\Delta v(x) + \rho h(x) e^{v(x)} = 4\pi \sum_{i=1}^{6} \delta_{P_i} - 2\pi \sum_{j=1}^{s} m_j (\delta_{Q_j} + \delta_{\iota(Q_j)}) \tag{1.3}$$

where $\iota : X \to X$ is an involution on X and ρ is some integer. The existence of the involution on X is due to the fact that any genus two compact Riemann surface is hyperelliptic. Let us formulate the precise statements of our main results as follows.

Let X be a compact Riemann surface[1] of genus $g \geq 2$ and ds^2 be any hermitian metric on X. The Gauss–Bonnet theorem tells us that

$$\int_X K d\nu = 2\pi\chi(X) = 4\pi(1-g) < 0,$$

where K is the Gaussian curvature function of (X, ds^2) and $d\nu$ is the volume form of (X, ds^2). Then the open set $U = \{P \in X : K(P) < 0\}$ is nonempty. Let $\mu : X \to \mathrm{Jac}(X)$ be the Abel–Jacobi map and $d\widetilde{s}^2$ be the standard flat metric on $\mathrm{Jac}(X)$. The canonical metric on X is the pull back metric $\mu^* d\widetilde{s}^2$. Now let ds^2 be the canonical metric on X and K be the corresponding Gaussian curvature function. It is not difficult to show that K is nonpositive on X, for example, see Section 2, or [4]. Let Z be the set of zeros of K. Then $U = X \setminus Z$. In [4], Lewittes shows that Z is nonempty if and only if X is a hyperelliptic Riemann surface and Z is the set of all Weierstrass points of X. As a consequence, the subset Z of X is either empty or a finite subset of X. If X is not a hyperelliptic Riemann surface, $U = X$. When X is a compact Riemann surface of genus two, X is hyperelliptic. Then we have the following result:

Theorem 1.1. *Let (X, ds^2) be a compact Riemann surface of genus $g = 2$ where ds^2 is the canonical metric on X. Define a function $u : X \setminus Z \to \mathbb{R}$ by $u = \log(-K)$, where $Z = \{P_1, \cdots, P_6\}$ is the set of all Weierstrass points of X. Then u is smooth on $X \setminus Z$ and satisfies the following mean field equation:*

$$\Delta u + 6e^u = 4\pi \sum_{i=1}^{6} \delta_{P_i}. \tag{1.4}$$

Here $\delta_P : C^\infty(X) \to \mathbb{R}$ is the Dirac delta function defined by $\delta_P(\varphi) = \varphi(P)$ for any $\varphi \in C^\infty(X)$ and for any $P \in X$.

[1] All the Riemann surfaces in this paper are assumed to be connected.

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