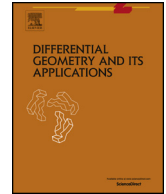


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## Four-dimensional homogeneous semi-symmetric Lorentzian manifolds

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## ABSTRACT

We determine all four-dimensional homogeneous semi-symmetric Lorentzian manifolds.

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## 1. Introduction

A pseudo-Riemannian manifold  $(M, g)$  is said to be semi-symmetric if its curvature tensor  $K$  satisfies  $K.K = 0$ . This is equivalent to

$$[K(X, Y), K(Z, T)] = K(K(X, Y)Z, T) + K(Z, K(X, Y)T), \quad (1)$$

for any vector fields  $X, Y, Z, T$ . Semi-symmetric pseudo-Riemannian manifolds generalize obviously locally symmetric manifolds ( $\nabla K = 0$ ). They also generalize second-order locally symmetric manifolds ( $\nabla^2 K = 0$  and  $\nabla K \neq 0$ ). Semi-symmetric Riemannian manifolds have been first investigated by E. Cartan [7] and the first example of a semi-symmetric not locally symmetric Riemannian manifold was given by Takagi [13]. More recently, Szabo [11,12] gave a complete description of these manifolds. In this study, Szabo used strong

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results proper to the Riemannian sitting which suggests that a similar study of semi-symmetric Lorentzian manifolds is far more difficult. To our knowledge, there are only few results on three dimensional locally homogeneous semi-symmetric Lorentzian manifolds [3,4] and second-order locally symmetric Lorentzian manifolds have been classified by D. Alekseevsky and A. Galaev in [1]. While in the Riemannian case every homogeneous semi-symmetric manifold is actually locally symmetric, in the Lorentzian case they are homogeneous semi-symmetric Lorentzian manifolds which are not locally symmetric.

This paper is devoted to the study of semi-symmetric curvature algebraic tensors on a Lorentzian vector space and to the classification of 4-dimensional simply-connected semi-symmetric homogeneous Lorentzian manifolds. There are our main results:

1. Let  $(V, \langle \cdot, \cdot \rangle)$  be a Lorentzian vector space and  $K : V \wedge V \rightarrow V \wedge V$  a semi-symmetric algebraic curvature tensor, i.e.,  $K$  satisfies the algebraic Bianchi identity and (1). Let  $\text{Ric}_K : V \rightarrow V$  be its Ricci operator. The main result here (see Propositions 2.1 and 2.2) is that  $\text{Ric}_K$  has only real eigenvalues and, if  $\lambda_1, \dots, \lambda_r$  are the non null ones then  $V$  splits orthogonally

$$V = V_0 \oplus V_{\lambda_1} \oplus \dots \oplus V_{\lambda_r}, \quad (2)$$

where  $V_{\lambda_i} = \ker(\text{Ric}_K - \lambda_i \text{Id}_V)$  and  $V_0 = \ker(\text{Ric}_K)^2$ . Moreover,  $\dim V_{\lambda_i} \geq 2$ ,  $K(V_{\lambda_i}, V_{\lambda_j}) = K(V_0, V_{\lambda_i}) = 0$  for  $i \neq j$ ,  $K(u, v)(V_{\lambda_i}) \subset V_{\lambda_i}$  and  $K(u, v)(V_0) \subset V_0$ . This reduces the study of semi-symmetric algebraic curvature tensors to the ones who are Einstein ( $\text{Ric}_K = \lambda \text{Id}_V$ ) or the ones who are Ricci isotropic ( $\text{Ric}_K \neq 0$  and  $(\text{Ric}_K)^2 = 0$ ).

2. In [8], Derdzinsky gave a classification of four dimensional Lorentzian Einstein manifolds whose curvature treated as a complex linear operator is diagonalizable and has constant eigenvalues. In [5], Calvaruso and Zaeim described locally homogeneous Lorentzian four-manifolds with diagonalizable Ricci operator. In [2], Astrakhantsev gave all semi-symmetric curvature tensors on a four dimensional Lorentzian vector space. Based on these three results, we prove the following two results.

**Theorem 1.1.** *Let  $M$  be a four-dimensional Einstein Lorentzian manifold with non null scalar curvature. Then  $M$  is semi-symmetric if and only if it is locally symmetric.*

**Theorem 1.2.** *Let  $M$  be a simply connected homogeneous semi-symmetric 4-dimensional Lorentzian manifold. If the Ricci tensor of  $M$  has a non zero eigenvalue then  $M$  is symmetric and in this case it is a product of a space of constant curvature and a Cahen–Wallace space.*

We start in Section 3 by proving Theorem 1.2 when  $M$  is Lie group endowed with a left invariant Lorentzian metric. In Section 4, we prove Theorems 1.1 and 1.2.

3. Having Theorem 1.2 in mind, to complete the classification of simply connected four-dimensional homogeneous semi-symmetric Lorentzian manifolds, we determine all simply connected four-dimensional semi-symmetric homogeneous Lorentzian manifolds with isotropic Ricci curvature. We will show that in this case  $(\text{Ric}_K)^2 = 0$  and  $K^2 = 0$ . To determine these spaces we distinguish two cases:
  - (a) Simply connected four-dimensional homogeneous semi-symmetric Lorentzian manifolds with non trivial isotropy and satisfying  $(\text{Ric}_K)^2 = 0$ . In Section 5, by using Komrakov's classification of four-dimensional homogeneous pseudo-Riemannian manifolds [9], we give the list of such spaces. In Theorem 5.1, we give the list of four-dimensional homogeneous semi-symmetric non symmetric Lorentzian manifolds with non trivial isotropy and which are Ricci flat. In Theorem 5.2, we give the list of four-dimensional homogeneous semi-symmetric non symmetric Lorentzian manifolds with non trivial isotropy and which are not Ricci flat. We point out that there are four-dimensional homogeneous symmetric Lorentzian manifolds which are Ricci isotropic even Ricci flat non flat (see Remark 2).

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