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Four-dimensional homogeneous semi-symmetric Lorentzian manifolds

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We determine all four-dimensional homogeneous semi-symmetric Lorentzian manifolds.

1. Introduction

A pseudo-Riemannian manifold (M, g) is said to be semi-symmetric if its curvature tensor K satisfies K.K $= 0$. This is equivalent to

$$[\mathrm{K}(X,Y), \mathrm{K}(Z,T)] = \mathrm{K}(\mathrm{K}(X,Y)Z,T) + \mathrm{K}(Z,\mathrm{K}(X,Y)T), \tag{1}$$

for any vector fields X, Y, Z, T. Semi-symmetric pseudo-Riemannian manifolds generalize obviously locally symmetric manifolds ($\nabla \mathrm{K} = 0$). They also generalize second-order locally symmetric manifolds ($\nabla^2\mathrm{K} = 0$ and $\nabla \mathrm{K} \neq 0$). Semi-symmetric Riemannian manifolds have been first investigated by E. Cartan [7] and the first example of a semi-symmetric not locally symmetric Riemannian manifold was given by Takagi [13]. More recently, Szabo [11,12] gave a complete description of these manifolds. In this study, Szabo used strong

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results proper to the Riemannian sitting which suggests that a similar study of semi-symmetric Lorentzian manifolds is far more difficult. To our knowledge, there are only few results on three dimensional locally homogeneous semi-symmetric Lorentzian manifolds [3,4] and second-order locally symmetric Lorentzian manifolds have been classified by D. Alekseevsky and A. Galaev in [1]. While in the Riemannian case every homogeneous semi-symmetric manifold is actually locally symmetric, in the Lorentzian case they are homogeneous semi-symmetric Lorentzian manifolds which are not locally symmetric.

This paper is devoted to the study of semi-symmetric curvature algebraic tensors on a Lorentzian vector space and to the classification of 4-dimensional simply-connected semi-symmetric homogeneous Lorentzian manifolds. There are our main results:

1. Let $(V, \langle\ ,\ \rangle)$ be a Lorentzian vector space and $\mathrm{K} : V \wedge V \longrightarrow V \wedge V$ a semi-symmetric algebraic curvature tensor, i.e., K satisfies the algebraic Bianchi identity and (1). Let $\mathrm{Ric}_K : V \longrightarrow V$ be its Ricci operator. The main result here (see Propositions 2.1 and 2.2) is that Ric_K has only real eigenvalues and, if $\lambda_1, \ldots, \lambda_r$ are the non null ones then V splits orthogonally

$$V = V_0 \oplus V_{\lambda_1} \oplus \ldots \oplus V_{\lambda_r}, \tag{2}$$

where $V_{\lambda_i} = \ker(\mathrm{Ric}_K - \lambda_i \mathrm{Id}_V)$ and $V_0 = \ker(\mathrm{Ric}_K)^2$. Moreover, $\dim V_{\lambda_i} \geq 2$, $\mathrm{K}(V_{\lambda_i}, V_{\lambda_j}) = \mathrm{K}(V_0, V_{\lambda_i}) = 0$ for $i \neq j$, $\mathrm{K}(u,v)(V_{\lambda_i}) \subset V_{\lambda_i}$ and $\mathrm{K}(u,v)(V_0) \subset V_0$. This reduces the study of semi-symmetric algebraic curvature tensors to the ones who are Einstein ($\mathrm{Ric}_K = \lambda \mathrm{Id}_V$) or the ones who are Ricci isotropic ($\mathrm{Ric}_K \neq 0$ and $(\mathrm{Ric}_K)^2 = 0$).
2. In [8], Derdzinsky gave a classification of four dimensional Lorentzian Einstein manifolds whose curvature treated as a complex linear operator is diagonalizable and has constant eigenvalues. In [5], Calvaruso and Zaeim described locally homogeneous Lorentzian four-manifolds with diagonalizable Ricci operator. In [2], Astrakhantsev gave all semi-symmetric curvature tensors on a four dimensional Lorentzian vector space. Based on these three results, we prove the following two results.

   Theorem 1.1. *Let M be a four-dimensional Einstein Lorentzian manifold with non null scalar curvature. Then M is semi-symmetric if and only if it is locally symmetric.*

   Theorem 1.2. *Let M be a simply connected homogeneous semi-symmetric 4-dimensional Lorentzian manifold. If the Ricci tensor of M has a non zero eigenvalue then M is symmetric and in this case it is a product of a space of constant curvature and a Cahen–Wallace space.*

   We start in Section 3 by proving Theorem 1.2 when M is Lie group endowed with a left invariant Lorentzian metric. In Section 4, we prove Theorems 1.1 and 1.2.
3. Having Theorem 1.2 in mind, to complete the classification of simply connected four-dimensional homogeneous semi-symmetric Lorentzian manifolds, we determine all simply connected four-dimensional semi-symmetric homogeneous Lorentzian manifolds with isotropic Ricci curvature. We will show that in this case $(\mathrm{Ric}_K)^2 = 0$ and $K^2 = 0$. To determine these spaces we distinguish two cases:
   (a) Simply connected four-dimensional homogeneous semi-symmetric Lorentzian manifolds with non trivial isotropy and satisfying $(\mathrm{Ric}_K)^2 = 0$. In Section 5, by using Komrakov's classification of four-dimensional homogeneous pseudo-Riemannian manifolds [9], we give the list of such spaces. In Theorem 5.1, we give the list of four-dimensional homogeneous semi-symmetric non symmetric Lorentzian manifolds with non trivial isotropy and which are Ricci flat. In Theorem 5.2, we give the list of four-dimensional homogeneous semi-symmetric non symmetric Lorentzian manifolds with non trivial isotropy and which are not Ricci flat. We point out that there are four-dimensional homogeneous symmetric Lorentzian manifolds which are Ricci isotropic even Ricci flat non flat (see Remark 2).