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The conformal vector fields on Kropina manifolds

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ABSTRACT

In this paper we study and characterize the conformal vector fields on Kropina manifolds. Furthermore, we obtain the explicit expressions of conformal vector fields on a Kropina manifold (M, F) of weakly isotropic flag curvature with the conditions that $b = \|\beta\|_\alpha$ is a constant and the dimension of M is greater than two.

1. Introduction

There is a special and important class of Finsler metrics in Finsler geometry which can be expressed in the form $F = \alpha\phi(\beta/\alpha)$, where α is a Riemannian metric and β is a 1-form and $\phi = \phi(s)$ is a C^∞ positive function on an open interval. We call this class of metrics the *(α, β)-metrics*. In the past several years, we witness a rapid development in Finsler geometry. This is partially due to the study on (α, β)-metrics ([2]). When $\phi = 1 + s$, the Finsler metric $F = \alpha + \beta$ is called *Randers metric*. When $\phi = 1/s$, the Finsler metric $F = \alpha^2/\beta$ is called *Kropina metric*. Randers metrics and Kropina metrics are both C-reducible ([10]). However, Randers metrics are regular Finsler metrics but Kropina metrics are Finsler metrics with singularity. Kropina metrics were first introduced by L. Berwald when he studied the two-dimensional Finsler spaces with rectilinear extremal and were investigated by V.K. Kropina (see [7][8]). Kropina metrics seem

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to be among the simplest nontrivial Finsler metrics with many interesting application in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics (see [1][6]). Also, they have interesting applications in relativistic field theory, control theory, evolution and developmental biology.

In this paper, we mainly study a class of Kropina metrics which are regular Finsler metrics on a manifold M. At each $x \in M$, we define

$$A_x = \{y = y^i \frac{\partial}{\partial x^i} \in T_xM \mid \beta = b_i(x)y^i > 0\},$$

which is a conic domain of T_xM whose boundary is the hyperplane $\{y = y^i \frac{\partial}{\partial x^i} \in T_xM \mid b_i(x)y^i = 0\}$. The Kropina metric $F = \alpha^2/\beta$: $A \to (0, \infty)$ is a conic Finsler metric which is positive definite Finsler metric on A, where $A := \bigcup_{x \in M} A_x$ (see [16]). This class of metrics are called the *conic Kropina metrics*. In the following, we just study conic Kropina metrics and we always use Kropina metric to take the place of conic Kropina metric.

The flag curvature in Finsler geometry is a natural analogue of sectional curvature in Riemannian geometry and is an important Riemannian geometric quantity. For a Finsler manifold (M, F), the flag curvature $\mathrm{K} = \mathrm{K}(P, y)$ of F is a function of “flag” $P \in T_xM$ and “flagpole” $y \in T_xM$ at x with $y \in P$. A Finsler metric F is said to be of *weakly isotropic flag curvature* if $\mathrm{K} = \frac{3\theta}{F} + \sigma$, where $\sigma = \sigma(x)$ is a scalar function and θ is a 1-form on M. In 1978, C. Shibata studied some basic local geometrical properties of Kropina spaces (see [11]). In 1991, M. Matsumoto obtained a set of necessary and sufficient conditions for a Kropina manifold to be of constant flag curvature (see [9]). Based on these results, R. Yoshikawa and K. Okubo characterized a Kropina metric $F = \frac{\alpha^2}{\beta}$ via navigation data (h, W), where h is a Riemannian metric and W is a unit length vector field, and obtained a minimal set of necessary and sufficient local conditions for a Kropina space to be of constant flag curvature (see [14,15]). Recently, Q. Xia has classified regular Kropina metrics of weakly isotropic flag curvature on the manifolds with the dimension greater than two under the condition that $b = \|\beta\|_\alpha$ is a constant (see [13]).

The conformal vector fields are the fundamental objects in differential geometry and physics. The conformal vector fields on a manifold M are vector fields induced by a local one-parameter group of conformal transformations of M. For Randers manifolds, based on the classification theorem on Randers metrics of weakly isotropic flag curvature given by the first author and Z. Shen in [3], Z. Shen and Q. Xia completely determined conformal vector fields on Randers manifolds of weakly isotropic flag curvature (see [12]). Then a natural problem arises: how to determine the conformal vector fields on Kropina manifolds of weakly isotropic flag curvature? In order to solve this problem, we first consider conformal vector fields on a Kropina manifold and obtain the following theorem.

Theorem 1.1. *Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on a smooth manifold M. Then a vector field V on (M, F) is a conformal vector field with conformal factor $\rho = \rho(x)$ if and only if V satisfies the following conditions:*

$$V_{i;j} + V_{j;i} = 4\lambda a_{ij}, \tag{1.1}$$

$$V^i b_{j;i} + b^i V_{i;j} = 2(2\lambda - \rho) b_j, \tag{1.2}$$

where we raise and lower the indices of V by a_{ij}, “;” denotes the covariant derivative with respect to Levi-Civita connection of Riemannian metric α and $\lambda = \lambda(x)$ is a scalar function on M.

Further, we will completely determine conformal vector fields from the PDEs in the Theorem 1.1 when F is a Kropina metric of weakly isotropic flag curvature and $b := \|\beta\|_\alpha$ is a constant.

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