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# Homogeneity for a class of Riemannian quotient manifolds

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ABSTRACT

We study Riemannian coverings $\varphi : \widetilde{M} \to \Gamma\backslash\widetilde{M}$ where $\widetilde{M}$ is a normal homogeneous space $G/K_1$ fibered over another normal homogeneous space $M = G/K$ and $K$ is locally isomorphic to a nontrivial product $K_1 \times K_2$. The most familiar such fibrations $\pi : \widetilde{M} \to M$ are the natural fibrations of Stiefel manifolds $SO(n_1+n_2)/SO(n_1)$ over Grassmann manifolds $SO(n_1+n_2)/[SO(n_1)\times SO(n_2)]$ and the twistor space bundles over quaternionic symmetric spaces (= quaternion-Kaehler symmetric spaces = Wolf spaces). The most familiar of these coverings $\varphi : \widetilde{M} \to \Gamma\backslash\widetilde{M}$ are the universal Riemannian coverings of spherical space forms. When $M = G/K$ is reasonably well understood, in particular when $G/K$ is a Riemannian symmetric space or when $K$ is a connected subgroup of maximal rank in $G$, we show that the Homogeneity Conjecture holds for $\widetilde{M}$. In other words we show that $\Gamma\backslash\widetilde{M}$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement. In order to find all the isometries of constant displacement on $\widetilde{M}$ we work out the full isometry group of $\widetilde{M}$, extending Élie Cartan's determination of the full group of isometries of a Riemannian symmetric space. We also discuss some pseudo-Riemannian extensions of our results.



## 1. Introduction

Some years ago I studied Riemannian covering spaces $S \to \Gamma\backslash S$ where $S$ is homogeneous. I conjectured that $\Gamma\backslash S$ is homogeneous if and only if every $\gamma \in \Gamma$ is an isometry of constant displacement (now usually called *Clifford translations* or *Clifford–Wolf isometries*) on $S$. I'll call that the *Homogeneity Conjecture*. This paper proves the conjecture for a class of normal Riemannian homogeneous spaces $\widetilde{M} = G/K_1$ that fiber over homogeneous spaces $M = G/K$ where $K_1$ is a local direct factor of $K$. The principal examples are those for which $K$ is the fixed point set of an automorphism of $G$ and the Lie algebra $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ with $\dim \mathfrak{k}_1 \neq 0 \neq \mathfrak{k}_2$. Those include the cases where $G/K$ is an hermitian symmetric space, or a Grassmann manifold of rank $> 1$, or a quaternion-Kaehler symmetric space (Wolf space), one of the irreducible nearly-Kaehler manifolds of $F_4$, $E_6$, $E_7$ or $E_8$, or everybody's favorite 5-symmetric space $E_8/A_4A_4$. See [35] and [36] for a complete list.

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For lack of a better term I'll refer to such spaces $\widetilde{M} = G/K_1$ as *isotropy–split* homogeneous spaces and to the fibration $\pi : \widetilde{M} \to M = G/K$ as an *isotropy–splitting* fibration.

Here we use isotropy–splitting fibrations $\pi : \widetilde{M} \to M$ as a bootstrap device to study Riemannian coverings $\varphi : \widetilde{M} \to \Gamma\backslash\widetilde{M}$. Specifically, $\pi$ is the projection given by $G/K_1 \to G/K$ with $K = K_1K_2$ where $M$ and $\widetilde{M}$ are normal Riemannian homogeneous spaces of the of the same group $G$ and each $\dim K_i > 0$. In particular, $\pi : \widetilde{M} \to M$ is a principal $K_2$-bundle. The point is to choose the splitting of $K$ so that $M$ is reasonably well understood. The most familiar example is the case where $\widetilde{M}$ is a Stiefel manifold and $M$ is the corresponding Grassmann manifold. More generally we study the situation where

$G$ is a compact connected simply connected Lie group,

$K = K_1K_2$ where the $K_i$ are closed connected subgroups of $G$ such that

$$\text{(i) } K = (K_1 \times K_2)/(K_1 \cap K_2), \text{ (ii) } \mathfrak{k}_2 \perp \mathfrak{k}_1 \text{ and } \text{ (iii) } \dim \mathfrak{k}_1 \neq 0 \neq \dim \mathfrak{k}_2\,, \tag{1.1}$$

the centralizers $Z_G(K_1) = Z_{K_1}\widetilde{K_2}$ and $Z_G(K_2) = Z_{K_2}\widetilde{K_1}$ with $K_1 = \widetilde{K_1}^0$ and $K_2 = \widetilde{K_2}^0$, and

$M = G/K$ and $\widetilde{M} = G/K_1$ are normal Riemannian homogeneous spaces of $G$.

Thus we may assume that the metrics on $M$ and $\widetilde{M}$ are the normal Riemannian metrics defined by the negative of the Killing form of $G$. Note that $\widetilde{M}$ and $M$ are simply connected, because $G$ is simply connected and $K_1$ and $K$ are connected.

**Lemma 1.2.** *There is no nonzero $G$-invariant vector field on $M$. In other words, if $\mathfrak{m} = \mathfrak{k}^\perp$ then $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})|_{\mathfrak{m}}$ has no nonzero fixed vector.*

**Proof.** The centralizer $Z_G(K)$ is finite by (1.1). □

Lemma 1.2 is of course obvious whenever $\mathrm{rank}\, K = \mathrm{rank}\, G$, in other words when the Euler characteristic $\chi(M) \neq 0$. The point here is that it holds as well when $\mathrm{rank}\, K < \mathrm{rank}\, G$.

Important examples of $M$ include the irreducible Riemannian symmetric spaces $G/K$ with $K$ not simple, the irreducible nearly-Kaehler manifolds of $F_4$, $E_6$, $E_7$ or $E_8$, and the very interesting 5-symmetric space $E_8/A_4A_4$. We will list these examples in detail and work out the precise structure of the group $I(\widetilde{M})$ of all isometries of $\widetilde{M}$. That is Theorem 3.12, and Corollary 3.5 identifies all the Killing vector fields on $\widetilde{M}$ of constant length. Killing vector fields of constant length are the infinitesimal version of isometries of constant displacement. After that we come to the main result, Theorem 5.6, which identifies all the isometries of constant displacement on $\widetilde{M}$. Applying it to a Riemannian covering $\widetilde{M} \to \Gamma\backslash\widetilde{M}$ we prove the Homogeneity Conjecture for isotropy–split manifolds. Then we sketch the mathematical background and current state for the Homogeneity Conjecture.

In Section 2 we view (compact) isotropy–splitting fibrations from the viewpoint of the Borel–de Siebenthal classification ([4], or see [32]) of pairs $(G, K)$ where $G$ is a compact connected simply connected simple Lie group and $K$ is a maximal subgroup of equal rank in $G$. This yields an explicit list. We then run through the cases where $G/K$ is a compact irreducible Riemannian symmetric space with $\mathrm{rank}\, K < \mathrm{rank}\, G$; the only ones that yield isotropy–splitting fibrations are the fibrations of real Stiefel manifolds over odd dimensional oriented real Grassmann manifolds. These are examples with which one can calculate explicitly, and to which our principal results apply.

In Section 3 we work out the full group of isometries of $\widetilde{M}$. The method combines ideas from Élie Cartan's description of the full isometry group of a Riemannian symmetric space, Carolyn Gordon's work on isometry groups of noncompact homogeneous spaces, and a theorem of Silvio Reggiani. The result is Theorem 3.12. One consequence, Corollary 3.5, is a complete description of the Killing vector fields of constant length on $\widetilde{M}$.