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On rank-critical matrix spaces

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ABSTRACT

A matrix space of size $m \times n$ is a linear subspace of the linear space of $m \times n$ matrices over a field $\mathbb{F}$. The rank of a matrix space is defined as the maximal rank over matrices in this space. A matrix space $\mathcal{A}$ is called rank-critical, if any matrix space which properly contains it has rank strictly greater than that of $\mathcal{A}$.

In this note, we first exhibit a necessary and sufficient condition for a matrix space $\mathcal{A}$ to be rank-critical, when $\mathbb{F}$ is large enough. This immediately implies the sufficient condition for a matrix space to be rank-critical by Draisma (2006) [5], albeit requiring the field to be slightly larger.

We then study rank-critical spaces in the context of compression and primitive matrix spaces. We first show that every rank-critical matrix space can be decomposed into a rank-critical compression matrix space and a rank-critical primitive matrix space. We then prove, using our necessary and sufficient condition, that the block-diagonal direct sum of two rank-critical matrix spaces is rank-critical if and only if both matrix spaces are primitive, when the field is large enough.

1. Results and discussions

1.1. *A necessary and sufficient condition for a matrix space to be rank-critical*

Let $\mathbb{F}$ be a field, and let $M(m \times n, \mathbb{F})$ be the linear space of $m \times n$ matrices over $\mathbb{F}$. A *matrix space* $\mathcal{A}$ is a linear subspace of $M(m \times n, \mathbb{F})$, denoted as $\mathcal{A} \leq M(m \times n, \mathbb{F})$. For $A \in M(m \times n, \mathbb{F})$ we denote its rank, kernel, and image, by $\mathrm{rk}(A)$, $\ker(A) \leq \mathbb{F}^n$, and $\mathrm{im}(A) \leq \mathbb{F}^m$, respectively. The *rank of a matrix space* $\mathcal{A}$, denoted as $\mathrm{rk}(\mathcal{A})$, is defined as $\max\{\mathrm{rk}(A) : A \in \mathcal{A}\}$. $\mathcal{A}$ is *singular*, if $\mathrm{rk}(\mathcal{A}) < \min(m, n)$. $\mathcal{A}$ is called *rank-critical*, if for any $\mathcal{B} \leq M(m \times n, \mathbb{F})$ with $\mathcal{B} \supsetneq \mathcal{A}$, $\mathrm{rk}(\mathcal{B}) > \mathrm{rk}(\mathcal{A})$. $GL(n, \mathbb{F})$ denotes the group of invertible $n \times n$ matrices over $\mathbb{F}$. Every $(g, h) \in GL(m, \mathbb{F}) \times GL(n, \mathbb{F})$ has a natural action on matrix spaces in $M(m \times n, \mathbb{F})$, by sending $\mathcal{A}$ to $g\mathcal{A}h^{-1}$. Two matrix spaces are equivalent if they are in the same orbit of this action.

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Our first result is a necessary and sufficient condition for a matrix space to be rank-critical. To state it, we introduce some notation. For $\mathcal{A} \leq M(m \times n, \mathbb{F})$, $\mathcal{A}_{\text{reg}} := \{A \in \mathcal{A} : \text{rk}(A) = \text{rk}(\mathcal{A})\}$. For two subspaces $U \leq \mathbb{F}^m$, $V \leq \mathbb{F}^n$ and $A \in M(m \times n, \mathbb{F})$, $A(V) = \{A(v) : v \in V\} \leq \mathbb{F}^m$ and $A^{-1}(U) = \{v \in \mathbb{F}^n : A(v) \in U\} \leq \mathbb{F}^n$. Note that the A^{-1} as in $A^{-1}(\cdot)$ does not refer to the inverse of A and A is not necessarily invertible.

The central notion in our condition is the following. Define the *rank neutral set* of $\mathcal{A} \leq M(m \times n, \mathbb{F})$ as

$$\text{RNS}(\mathcal{A}) := \{B \in M(m \times n, \mathbb{F}) : \forall A \in \mathcal{A}_{\text{reg}}, \forall k \in \{0, 1, \ldots, m\}, B(A^{-1}B)^k \ker(A) \subseteq \text{im}(A)\}. \tag{1}$$

The elements of $\text{RNS}(\mathcal{A})$ are called the *rank neutral elements* of $\mathcal{A}$. Note that for $(g, h) \in GL(m, \mathbb{F}) \times GL(n, \mathbb{F})$, $\text{RNS}(g\mathcal{A}h^{-1}) = g\text{RNS}(\mathcal{A})h^{-1}$.

Theorem 1. *Let $\mathcal{A} \leq M(m \times n, \mathbb{F})$ and suppose $|\mathbb{F}| \geq 2 \cdot \min(m, n)$. Then $\text{RNS}(\mathcal{A}) \supseteq \mathcal{A}$, and $\mathcal{A}$ is rank-critical if and only if $\text{RNS}(\mathcal{A}) = \mathcal{A}$. Furthermore, given $G \leq GL(m, \mathbb{F}) \times GL(n, \mathbb{F})$ with the natural action on matrix spaces, if $\mathcal{A}$ is stable under G, then $\text{RNS}(\mathcal{A})$ is also stable under G.*

We deduce the sufficient condition for a matrix space to be rank-critical by Draisma [5], which plays a key role there to prove that the images of certain Lie algebra representations are rank-critical. The key notion in Draisma's condition is the set of *rank neural directions* of $\mathcal{A} \leq M(m \times n, \mathbb{F})$,

$$\text{RND}(\mathcal{A}) := \{B \in M(m \times n, \mathbb{F}) : \forall A \in \mathcal{A}_{\text{reg}}, B \ker(A) \subseteq \text{im}(A)\}.$$

Clearly, $\text{RND}(\mathcal{A}) \supseteq \text{RNS}(\mathcal{A})$. Furthermore if a group action is present as described in Theorem 1, then $\text{RND}(\mathcal{A})$ is also a stable set under the action of G. Therefore the following result by Draisma follows immediately from Theorem 1.

Corollary 2 *([5, Proposition 3]). Let $\mathcal{A} \leq M(m \times n, \mathbb{F})$ and suppose $|\mathbb{F}| \geq 2 \cdot \min(n, m)$. Then $\text{RND}(\mathcal{A}) \supseteq \mathcal{A}$, and if $\text{RND}(\mathcal{A}) = \mathcal{A}$ then $\mathcal{A}$ is rank-critical. Furthermore, given $G \leq GL(m, \mathbb{F}) \times GL(n, \mathbb{F})$ with the natural action on matrix spaces, if $\mathcal{A}$ is stable under G, then $\text{RND}(\mathcal{A})$ is also stable under G.*

We note the following differences between Corollary 2 and [5, Prop. 3], though such differences are mostly superficial. On one hand, Corollary 2 requires the field to be slightly larger than needed in [5, Prop. 3]: there it only requires $|\mathbb{F}| > \text{rk}(\mathcal{A})$ (see also Remark 9). On the other hand, Corollary 2 deals with matrix spaces that are not necessarily square, and handles a more general group action.

1.2. Discussion: about the discrepancy between RND *and* RNS

In [5], Draisma asked the question to investigate the "discrepancy between rank-criticality and $\mathcal{A} = \text{RND}(\mathcal{A})$." Theorem 1 may be used as a guide to answer this question: it is now enough to investigate the discrepancy between $\text{RNS}(\mathcal{A})$ and $\text{RND}(\mathcal{A})$.

Suppose that we want to prove that there exist families of rank-critical matrix spaces $\mathcal{A}$ satisfying $\text{RNS}(\mathcal{A}) \subsetneq \text{RND}(\mathcal{A})$, by constructing an explicit example. This would require us to examine those rank-critical matrix spaces whose rank-criticality is not proved using the $\text{RND}(\mathcal{A}) = \mathcal{A}$ condition. However, we know few examples satisfying this criterion. The only such family of matrix spaces we are aware of is derived from linear maps of the form $\wedge^k V \to \wedge^{k+1} V$, consisting of those sending $s \in \wedge^k V \to s \wedge t \in \wedge^{k+1} V$ over all $t \in V$ (see e.g. [8,15,3]). However, we checked some minimal examples, including the two 4×4 spaces from [7, Theorem 1.2], and the 5-dimensional space of 10×10 matrices consisting of those sending $s \in \wedge^2 \mathbb{F}^5$ to $s \wedge t \in \wedge^3 \mathbb{F}^5$ over all $t \in \mathbb{F}^5$, and found that they satisfy $\text{RND}(\mathcal{A}) = \mathcal{A}$.

It is then desirable to examine those rank-critical matrix spaces that are assembled from known rank-critical matrix spaces. A basic structural result about matrix spaces of rank bounded from above is based on

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