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Tensor decomposition and homotopy continuation

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A B S T R A C T

A computationally challenging classical elimination theory problem is to compute polynomials which vanish on the set of tensors of a given rank. By moving away from computing polynomials via elimination theory to computing pseudowitness sets via numerical elimination theory, we develop computational methods for computing ranks and border ranks of tensors along with decompositions. More generally, we present our approach using joins of any collection of irreducible and nondegenerate projective varieties $X_1, \ldots, X_k \subset \mathbb{P}^N$ defined over $\mathbb{C}$. After computing ranks over $\mathbb{C}$, we also explore computing real ranks. A variety of examples are included to demonstrate the numerical algebraic geometric approaches.

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0. Introduction

Computing tensor decompositions is a fundamental problem in numerous application areas including computational complexity, signal processing for telecommunications [30,40], scientific data analysis [58,75], electrical engineering [27], and statistics [65]. Some other applications include the complexity of matrix multiplication [81], the complexity problem of P versus NP [83], the study of entanglement in quantum physics [42], matchgates in computer science [83], the study of phylogenetic invariants [6], independent component analysis [29], blind identification in signal processing [74], branching structure in diffusion images [72], and other multilinear data analytic techniques in bioinformatics and spectroscopy [31].

One computational algebraic geometric approach for deciding if a decomposition can exist is to compute equations that define secant and join varieties (e.g., see [60, Chap. 7] for a general overview). This can be formulated as a classical elimination theory question which, at least in theory, can be computed using Gröbner basis methods. Moreover, the defining equations do not yield decompositions, only existential information. Rather than focusing on computing defining equations, this paper uses numerical algebraic geometry (e.g., see [12,80] for a general overview) for performing membership tests and computing decompositions. In particular, we use *numerical elimination theory* to perform the computations based on the methods developed in [52,53] (see also [12, Chap. 16]). This approach differs from several previous methods of combining numerical algebraic geometry and elimination theory, e.g., [10, § 3.3–3.4] and [78,79], in that these previous methods relied upon interpolation.

The general setup for this paper is as follows. Let $X \subset \mathbb{P}^N$ be an irreducible and nondegenerate projective variety defined over $\mathbb{C}$ and $\mathcal{C}(X) \subset \mathbb{C}^{N+1}$ be the affine cone of X. We let a point P be a nonzero vector in $\mathbb{C}^{N+1}$ while $[P]$ denotes the line in $\mathbb{C}^{N+1}$ passing through the origin and P, i.e., $[P] \in \mathbb{P}^N$ is the projectivization of $P \in \mathbb{C}^{N+1}$. The *X-rank* of $[P] \in \mathbb{P}^N$ (or of $P \in \mathbb{C}^{N+1}$), denoted $\mathrm{rk}_X(P)$, is the minimum $r \in \mathbb{N}$ such that P can be written as a linear combination of r elements of $\mathcal{C}(X)$:

$$P = \sum_{i=1}^{r} x_i, \quad x_i \in \mathcal{C}(X). \tag{1}$$

Let $\sigma_r^0(X) \subset \mathbb{P}^N$ denote the set of elements with rank at most r and, for $[x_i] \in \mathbb{P}^N$, let $\langle [x_1], \ldots, [x_r] \rangle$ denote the linear space spanned by $x_1, \ldots, x_r$. The *r-th secant variety* of X is

$$\sigma_r(X) = \overline{\sigma_r^0(X)} = \overline{\bigcup_{[x_1],\ldots,[x_r] \in X} \langle [x_1], \ldots, [x_r] \rangle}.$$

In particular, if $[P] \in \sigma_r(X)$, then $[P]$ is the limit of a sequence of elements of X-rank at most r. The *X-border rank* of $[P]$, denoted $\mathrm{brk}_X(P)$, is the minimum $r \in \mathbb{N}$ such that $[P] \in \sigma_r(X)$. Obviously, $\mathrm{brk}_X(P) \le \mathrm{rk}_X(P)$.

Secant varieties are just special cases of join varieties. For irreducible and nondegenerate projective varieties $X_1, \ldots, X_k$, the constructible join and join variety of $X_1, \ldots, X_k$, respectively, are

$$J^0(X_1, \ldots, X_k) = \bigcup_{[x_1] \in X_1, \ldots, [x_k] \in X_k} \langle [x_1], \ldots, [x_k] \rangle \quad \text{and} \quad J(X_1, \ldots, X_k) = \overline{J^0(X_1, \ldots, X_k)}. \tag{2}$$

Clearly, $\sigma_r^0(X) = J^0(\underbrace{X, \ldots, X}_{r})$ and $\sigma_r(X) = J(\underbrace{X, \ldots, X}_{r})$.

As mentioned above, one can test, in principle, if an element belongs to a certain join variety (or if it has certain X-border rank) by computing defining equations for the join variety (or the secant variety, respectively). Unfortunately, finding defining equations for secant and join varieties is generally a very difficult elimination problem which is far from being well understood at this time.

The following summarizes the remaining sections of this paper.

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