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Geometric complexity theory and matrix powering

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ABSTRACT

Valiant's famous determinant versus permanent problem is the flagship problem in algebraic complexity theory. Mulmuley and Sohoni (2001, 2008) [23,24] introduced geometric complexity theory, an approach to study this and related problems via algebraic geometry and representation theory. Their approach works by multiplying the permanent polynomial with a high power of a linear form (a process called padding) and then comparing the orbit closures of the determinant and the padded permanent. This padding was recently used heavily to show negative results for the method of shifted partial derivatives (Efremenko et al., 2016 [6]) and for geometric complexity theory (Ikenmeyer and Panova, 2016 [17] and Bürgisser et al., 2016 [3]), in which occurrence obstructions were ruled out to be able to prove superpolynomial complexity lower bounds. Following a classical homogenization result of Nisan (1991) [25] we replace the determinant in geometric complexity theory with the trace of a symbolic matrix power. This gives an equivalent but much cleaner homogeneous formulation of geometric complexity theory in which the padding is removed. This radically changes the representation theoretic questions involved to prove complexity lower bounds. We prove that in this homogeneous formulation there are no orbit occurrence obstructions that prove even superlinear lower bounds on the complexity of the permanent.

Interestingly—in contrast to the determinant—the trace of a symbolic matrix power is not uniquely determined by its stabilizer.

1. Statement of the result

Let $\mathrm{per}_m := \sum_{\sigma \in \mathfrak{S}_m} \prod_{i=1}^m X_{i,\sigma(i)}$ denote the $m \times m$ permanent polynomial and let $\mathrm{Pow}_n^m := \mathrm{tr}(X^m)$ denote the trace of the mth power of an $n \times n$ matrix $X = (X_{i,j})$ of variables. The coordinate rings of the orbits and orbit closures $\mathbb{C}[\mathsf{GL}_{n^2}\mathrm{Pow}_n^m]$ and $\mathbb{C}[\overline{\mathsf{GL}_{n^2}\mathrm{per}_m}]$ are GL_{n^2}-representations. Let λ be an isomorphism

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type of an irreducible GL_{n^2}-representation. In this paper we prove that if $n \geq m+2 \geq 12$ and λ occurs in $\mathbb{C}[\overline{\mathsf{GL}_{n^2}\mathrm{per}_m}]$, then λ also occurs in $\mathbb{C}[\mathsf{GL}_{n^2}\mathrm{Pow}_n^m]$, see Theorem 2.12 below.

2. Introduction

Valiant's famous determinant versus permanent problem is a major open problem in computational complexity theory. It can be stated as follows, see Conjecture 2.1: For a polynomial p in any number of variables let the determinantal complexity $\mathsf{dc}(p)$ denote the smallest $n \in \mathbb{N}$ such that p can be written as the determinant $p = \det(A)$ of an $n \times n$ matrix A whose entries are affine linear forms in the variables.

Throughout the paper we fix our ground field to be the complex numbers $\mathbb{C}$. The permanent is of interest in combinatorics and theoretical physics, but our main interest stems from the fact that it is complete for the complexity class **VNP** (although the arguments in this paper remain valid if the permanent is replaced by any other **VNP**-complete function, mutatis mutandis). Valiant famously posed the following conjecture.

Conjecture 2.1. *The sequence* $\mathsf{dc}(\mathrm{per}_m)$ *grows superpolynomially.*

Valiant [31] proved that Conjecture 2.1 implies the separation $\mathbf{VP}_{\mathrm{e}} \subsetneq \mathbf{VNP}$ of algebraic complexity classes, which was later refined in [30], see also [21]: Conjecture 2.1 is equivalent to the separation $\mathbf{VP}_{\mathrm{s}} \subsetneq \mathbf{VNP}$. Many polynomially equivalent formulations for the determinantal complexity exist. For example $\mathsf{dc}(p)$ is polynomially equivalent to the smallest size of a skew circuit computing p, or the smallest size of a weakly skew circuit computing p, or the smallest size of an algebraic branching program computing p.

2.1. Preliminaries and the padded setting

Geometric complexity theory was introduced by Mulmuley and Sohoni [23,24] to resolve Conjecture 2.1 and related conjectures as follows. For $n > m$ define the *padded permanent* $\mathrm{per}_m^n := (X_{n,n})^{n-m}\mathrm{per}_m$, which is homogeneous of degree n in m^2+1 variables. Let $\mathbb{A}_n$ denote the vector space of homogeneous degree n polynomials in the n^2 variables $X_{i,j}$. Clearly $\mathrm{per}_m^n \in \mathbb{A}_n$. Moreover, $\det_n \in \mathbb{A}_n$, where $\det_n := \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma)\prod_{i=1}^n X_{i,\sigma(i)}$ is the determinant polynomial. The group GL_{n^2} of invertible $n^2 \times n^2$ matrices acts canonically on $\mathbb{A}_n$ by replacing variables with homogeneous linear forms. Let $\mathsf{GL}_{n^2}\det_n := \{g \cdot \det_n \mid g \in \mathsf{GL}_{n^2}\}$ be the orbit of the determinant and analogously let $\mathsf{GL}_{n^2}\mathrm{per}_m^n$ be the orbit of the padded permanent. Let $\overline{\mathsf{GL}_{n^2}\det_n} \subseteq \mathbb{A}_n$ and $\overline{\mathsf{GL}_{n^2}\mathrm{per}_m^n} \subseteq \mathbb{A}_n$ denote the closures of the respective orbits in $\mathbb{A}_n$. Here Euclidean closure and Zariski closure coincide [19, II.2.2 c & AI.7.2 Folgerung], i.e., both orbit closures are affine subvarieties of $\mathbb{A}_n$. Mulmuley and Sohoni proposed the following way to find lower bounds on $\mathsf{dc}(\mathrm{per}_m)$.

Proposition 2.2. *If* $\overline{\mathsf{GL}_{n^2}\mathrm{per}_m^n} \not\subseteq \overline{\mathsf{GL}_{n^2}\det_n}$, *then* $\mathsf{dc}(\mathrm{per}_m) > n$.

We call Proposition 2.2 the *padded setting*. To prove lower bounds on $\mathsf{dc}(\mathrm{per}_m)$ Mulmuley and Sohoni [23,24] suggested to study the representation theory of the coordinate rings of the orbits and orbit closures and use so-called *occurrence obstructions*. To define occurrence obstructions we now discuss the representation theory of the coordinate rings.

Recall that $\mathbb{A}_n$ is a complex vector space of dimension $\binom{n^2+n-1}{n}$. Let $\mathbb{C}[\mathbb{A}_n]$ denote its coordinate ring, i.e., the ring of polynomials in $\binom{n^2+n-1}{n}$ variables. The group GL_{n^2} acts linearly in a canonical way on each homogeneous degree d component $\mathbb{C}[\mathbb{A}_n]_d$ by the canonical pullback $(gf)(p) := f(g^{-1}p)$, for all $f \in \mathbb{C}[\mathbb{A}_n]_d$, $g \in \mathsf{GL}_{n^2}$, $p \in \mathbb{A}_n$. Since the group GL_{n^2} is reductive, the finite dimensional GL_{n^2}-representation $\mathbb{C}[\mathbb{A}_n]_d$ splits into a direct sum representations: $\mathbb{C}[\mathbb{A}_n]_d = \bigoplus_i V_i$, where each V_i is an irreducible GL_{n^2}-representation, i.e., a vector space with no nontrivial linear subspaces that are invariant under the group action. Two irreducible GL_{n^2}-representations V_i and V_j are called *isomorphic* if there exists a GL_{n^2}-equivariant vector

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