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Permanent v. determinant: An exponential lower bound assuming symmetry and a potential path towards Valiant's conjecture

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ABSTRACT

We initiate a study of determinantal representations with symmetry. We show that Grenet's determinantal representation for the permanent is optimal among determinantal representations equivariant with respect to left multiplication by permutation and diagonal matrices (roughly half the symmetry group of the permanent). We introduce a restricted model of computation, *equivariant determinantal complexity*, and prove an exponential separation of the permanent and the determinant in this model. This is the first exponential separation of the permanent from the determinant in any restricted model.

1. Introduction

Perhaps the most studied polynomial of all is the determinant:

$$\det_n(x) := \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) x^1_{\sigma(1)} x^2_{\sigma(2)} \cdots x^n_{\sigma(n)}, \tag{1}$$

a homogeneous polynomial of degree n in n^2 variables. Here $\mathfrak{S}_n$ denotes the group of permutations on n elements and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ.

Despite its formula with $n!$ terms, $\det_n$ can be evaluated quickly, e.g., using Gaussian elimination, which exploits the large symmetry group of the determinant, e.g., $\det_n(x) = \det_n(AxB^{-1})$ for any $n \times n$ matrices A, B with determinant equal to one.

We will work exclusively over the complex numbers and with homogeneous polynomials, the latter restriction only for convenience. L. Valiant showed in [26] that given a homogeneous polynomial $P(y)$ in M

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variables, there exists an n and an affine linear map $\tilde{A}:\mathbb{C}^M\to\mathbb{C}^{n^2}$ such that $P=\det_n\circ\tilde{A}$. Such $\tilde{A}$ is called a *determinantal representation* of P. When $M=m^2$ and P is the permanent polynomial

$$\mathrm{perm}_m(y):=\sum_{\sigma\in\mathfrak{S}_m} y^1_{\sigma(1)}y^2_{\sigma(2)}\cdots y^m_{\sigma(m)}, \tag{2}$$

he showed that one can take $n=O(2^m)$. As an algebraic analog of the $\mathbf{P}\neq\mathbf{NP}$ conjecture, he also conjectured that one cannot do much better:

Conjecture 1.1 *(Valiant [27]). Let $n(m)$ be a function of m such that there exist affine linear maps $\tilde{A}_m:\mathbb{C}^{m^2}\to\mathbb{C}^{n(m)^2}$ satisfying*

$$\mathrm{perm}_m=\det_{n(m)}\circ\tilde{A}_m. \tag{3}$$

Then $n(m)$ grows faster than any polynomial in m.

To measure progress towards Conjecture 1.1, define $\mathrm{dc}(\mathrm{perm}_m)$ to be the smallest $n(m)$ such that there exists $\tilde{A}_m$ satisfying (3). The conjecture is that $\mathrm{dc}(\mathrm{perm}_m)$ grows faster than any polynomial in m. Lower bounds on $\mathrm{dc}(\mathrm{perm}_m)$ are: $\mathrm{dc}(\mathrm{perm}_m)>m$ (Marcus and Minc [16]), $\mathrm{dc}(\mathrm{perm}_m)>1.06m$ (Von zur Gathen [29]), $\mathrm{dc}(\mathrm{perm}_m)>\sqrt{2}m-O(\sqrt{m})$ (Meshulam, reported in [29], and Cai [5]), with the current world record $\mathrm{dc}(\mathrm{perm}_m)\geq\frac{m^2}{2}$ [19] by Mignon and the second author. (Over $\mathbb{R}$, Yabe recently showed that $\mathrm{dc}_{\mathbb{R}}(\mathrm{perm}_m)\geq m^2-2m+2$ [30], and in [6] Cai, Chen and Li extended the $\frac{m^2}{2}$ bound to arbitrary fields.)

Inspired by *Geometric Complexity Theory* (GCT) [20], we focus on the *symmetries* of $\det_n$ and perm_m. Let V be a complex vector space of dimension M, let $\mathrm{GL}(V)$ denote the group of invertible linear maps $V\to V$. For $P\in S^mV^*$, a homogeneous polynomial of degree m on V, let

$$G_P:=\{g\in\mathrm{GL}(V)\mid P(g^{-1}y)=P(y)\quad\forall y\in V\}$$
$$\mathbb{G}_P:=\{g\in\mathrm{GL}(V)\mid P(g^{-1}y)\in\mathbb{C}^*P(y)\quad\forall y\in V\}$$

denote the *symmetry group (resp. projective symmetry group)* of P. The function $\chi_P:\mathbb{G}_P\to\mathbb{C}^*$ defined by the equality $P(g^{-1}y)=\chi_P(g)P(y)$ is group homomorphism called the *character* of P. For example $\mathbb{G}_{\det_n}\simeq(\mathrm{GL}_n\times\mathrm{GL}_n)/\mathbb{C}^*\rtimes\mathbb{Z}_2$ [9], where the $\mathrm{GL}_n\times\mathrm{GL}_n$ invariance comes from $\det_n(AxB^{-1})=(\det_n A\det_n B^{-1})\det_n(x)$ and the $\mathbb{Z}_2$ is because $\det_n(x)=\det_n(x^T)$ where x^T is the transpose of the matrix x. Write $\tau:\mathrm{GL}_n\times\mathrm{GL}_n\to\mathrm{GL}_{n^2}$ for the map $(A,B)\mapsto\{x\mapsto AxB^{-1}\}$. The character $\chi_{\det_n}$ satisfies $\chi_{\det_n}\circ\tau(A,B)=\det(A)\det(B)^{-1}$.

As observed in [20], the permanent (resp. determinant) is *characterized by its symmetries* and its degree in the sense that any polynomial $P\in S^m\mathbb{C}^{m^2*}$ with a symmetry group G_P such that $G_P\supseteq G_{\mathrm{perm}_m}$ (resp. $G_P\supseteq G_{\det_m}$) is a scalar multiple of the permanent (resp. determinant). This property is the cornerstone of GCT. The program outlined in [20,21] is an approach to Valiant's conjecture based on the functions on GL_{n^2} that respect the symmetry group $G_{\det_n}$, i.e., are invariant under the action of $G_{\det_n}$.

The interest in considering $\mathbb{G}_P$ instead of G_P is that if P is characterized by G_P among homogeneous polynomials of the same degree, then it is characterized by the pair $(\mathbb{G}_P,\chi_P)$ among all polynomials. This will be useful, since *a priori*, $\det_n\circ\tilde{A}$ need not be homogeneous.

Guided by the principles of GCT, we ask:

What are the $\tilde{A}$ that respect the symmetry group of the permanent?