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On four-dimensional Einstein affine hyperspheres ☆

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ABSTRACT

It is well-known that Vrancken–Li–Simon classified locally strongly convex affine hyperspheres in $\mathbb{R}^{n+1}$ whose affine metric are of constant sectional curvatures, but on the other side it is still a difficult problem to classify n-dimensional locally strongly convex affine hyperspheres whose affine metrics are Einstein. In this paper, we have solved the problem in case $n = 4$.

1. Introduction

Considering affine hypersurfaces, we denote by $\mathbb{R}^{n+1}$ the real unimodular-affine space equipped with its canonical flat connection D and a parallel volume form ω. Let $M := M^n$ be a differentiable, connected C^∞-manifold of dimension $n \geq 2$, and let $F : M^n \to \mathbb{R}^{n+1}$ be a non-degenerate hypersurface immersion with equiaffine (unimodular) normal ξ ([15]). We denote by h its affine Blaschke–Berwald metric which is semi-Riemannian, by S the affine shape operator and by ∇ its induced affine connection. Let $\hat{\nabla}$ be the Levi-Civita connection of the affine metric h. The difference tensor K is defined by $K(X, Y) := K_X Y := \nabla_X Y - \hat{\nabla}_X Y$; it is symmetric as both connections are torsion free.

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Here in this paper, we will always assume that the hypersurface is locally strongly convex, i.e., the affine metric is definite. Then, if necessary, by changing the sign of the affine normal, we may always assume that the affine metric is positive definite.

The hypersurface is called an affine hypersphere if $S = L_1 \cdot \mathrm{id}$. In that case, one easily proves the affine mean curvature $\frac{1}{n}\,\mathrm{trace}\,S = L_1 = \mathrm{const}$. More precisely, F is called a proper affine hypersphere if $L_1 \neq 0$; if $L_1 > 0$ (resp. $L_1 < 0$), the proper affine hypersphere is called elliptic (resp. hyperbolic). If $L_1 = 0$, the affine hypersphere is called improper or parabolic. For a proper affine hypersphere the affine normal satisfies $\xi(p) = -L_1\,(F(p) - c)$, where c is a constant vector, called the center of $F(M^n)$; for simplicity, we choose c as origin. For an improper affine hypersphere the affine normal field is constant.

The affine hyperspheres form a very important class of affine hypersurfaces. From a global point of view locally strongly convex hyperbolic affine hyperspheres have been widely studied, see amongst others the works of [3,9,10,12,13,17] or the book of A.-M. Li, U. Simon, G.S. Zhao and Z.J. Hu [14], also see the recent survey paper [19]. Even assuming globally conditions, the class of hyperbolic affine hyperspheres is surprisingly large. Even more, locally, in arbitrary dimensions one is still far away from a complete understanding of such hypersurfaces.

Worthwhile to mention from a local point of view are the classification of the affine hyperspheres with constant sectional curvature, see [11,22] for the locally strongly convex case, or [20,21] for the general non-degenerate case and the Calabi construction ([2,4]) of hyperbolic affine hyperspheres which allows to associate with two hyperbolic affine hyperspheres $\psi_1 : M_1^{n_1} \to \mathbb{R}^{n_1+1}$ and $\psi_2 : M_2^{n_2} \to \mathbb{R}^{n_2+1}$, two new immersions: φ and $\tilde{\varphi}$: for $p \in M_1, t \in \mathbb{R}$,

$$\varphi(p,t) = (c_1 e^{\frac{t}{\sqrt{n_1+1}}}\psi_1(p), c_2 e^{-\sqrt{n_1+1}t}) \in \mathbb{R}^{n_1+2},$$

and, for $p \in M_1^{n_1}, q \in M_2^{n_2}, t \in \mathbb{R}$,

$$\tilde{\varphi}(p,q,t) = (c_1 e^{\sqrt{\frac{n_2+1}{n_1+1}}\,t}\psi_1(p), c_2 e^{-\sqrt{\frac{n_1+1}{n_2+1}}\,t}\psi_2(q)) \in \mathbb{R}^{n_1+n_2+2},$$

which are both again hyperbolic affine hyperspheres. Here, φ and $\tilde{\varphi}$ are respectively called the Calabi product of an affine hypersphere and a point, and the Calabi product of two hyperbolic affine hyperspheres. Note that characterizations of these Calabi products are established in [6] which becomes a crucial step for the complete classification of locally strongly convex affine hypersurfaces with parallel difference tensor (i.e. cubic form) [7].

After having the classification of the affine hyperspheres with constant sectional curvature ([11,22]) and the classification of the affine hypersphere with parallel cubic form ([5–8]), the next important problem becomes natural and interesting:

Problem. Classify all n-dimensional locally strongly convex affine hyperspheres whose affine metrics are Einstein in $\mathbb{R}^{n+1}$.

In this paper, we have solved the above problem in case $n = 4$, that is, we have proved the following theorem:

Theorem. *Let $x : M^4 \hookrightarrow \mathbb{R}^5$ be a locally strongly convex affine hypersphere with its affine metric being Einstein, then it is locally affine equivalent to the open part of either one of the hyperquadrics, or the hyperbolic affine hypersphere $Q(1,4):\ x_1x_2x_3x_4x_5 = 1$, where $(x_1, \cdots, x_5)$ are the coordinates of $\mathbb{R}^5$.*

This paper is organized as follows. In Section 2, we briefly recall the theory of local affine hypersurfaces. In Section 3, we review the construction of a typical orthonormal basis at a fixed point, by which we need to