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Natural operations on differential forms on contact manifolds

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ABSTRACT

We characterize all natural linear operations between spaces of differential forms on contact manifolds. Our main theorem says roughly that such operations are built from some algebraic operators which we introduce and the exterior derivative.

1. Introduction

A classical theorem due to Palais [20] characterizes those linear operations on differential forms on a manifold which are compatible with diffeomorphisms. The result is roughly that only the identity and the exterior differential have these properties. The linearity assumption was removed by Kolář–Michor–Slovák in 1993 [18]. A very recent result by Navarro–Sancho [19] generalizes this theorem further by considering natural operations on k-tuples of differential forms. They prove that such operations can be written as polynomials in the given forms and their exterior differentials. A special case of this theorem was shown earlier by Freed–Hopkins [13].

A natural general question in this context is the following. Assume that M is endowed with some extra structure (for instance, a contact or symplectic structure, an almost complex structure etc.). What are the operations on differential forms which are compatible with the diffeomorphisms of M respecting the extra structure? In the present paper, we study this question for contact manifolds and contactomorphisms.

Before describing more precisely our result, let us recall the definition of contact manifolds, and the construction of the Rumin differential operator.

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A smooth manifold M of dimension $2n+1$, endowed with a completely non-integrable field Q of hyperplanes (called contact planes) is called a contact manifold. We refer to [15] for contact manifolds and to [11,17] for some of the symplectic linear algebra which we will use.

Locally, the contact plane can be written as $\ker\alpha$ for some 1-form α. The condition of complete non-integrability is that $\alpha\wedge(\mathrm{d}\alpha)^n \neq 0$. A contactomorphism (resp. local contactomorphism) $\phi:(M,Q)\to(\tilde{M},\tilde{Q})$ is a diffeomorphism (resp. local diffeomorphism) such that $\mathrm{d}\phi(Q)=\tilde{Q}$. If $Q=\ker\alpha, \tilde{Q}=\ker\tilde{\alpha}$, then a (local) diffeomorphism ϕ is a (local) contactomorphism if and only if $\phi^*\tilde{\alpha}=f\alpha$ for some nowhere vanishing function f. If $f\equiv 1$, ϕ is called a strict contactomorphism. The group of all contactomorphisms of (M,Q) is denoted by $\mathrm{Cont}(M)$.

The standard example of a contact manifold is $\mathbb{R}^{2n+1}$ with coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n,z)$ and $\alpha:=\mathrm{d}z+\sum_{i=1}^n x_i\mathrm{d}y_i$. Darboux' theorem states that locally each contact manifold of dimension $2n+1$ is contactomorphic to $\mathbb{R}^{2n+1}$ with its standard contact structure.

Let $\Omega^a(M)$ denote the space of differential a-forms on M. Rumin constructed an operator $\mathrm{D}:\Omega^n(M)\to\Omega^{n+1}(M)$ as follows. Let $\omega\in\Omega^n(M)$. Restrict $\mathrm{d}\omega$ to the contact plane. By some basic linear symplectic algebra, we may write $\mathrm{d}\omega|_Q=-d\alpha\wedge\xi|_Q$, where $\xi\in\Omega^{n-1}(M)$. Moreover, $\alpha\wedge\xi$ is unique and independent of the choice of the local defining 1-form α. Then

$$\mathrm{D}(\omega):=\mathrm{d}(\omega+\alpha\wedge\xi)$$

is well-defined. The second order differential operator D is called Rumin operator. Another natural operator is given by $\mathrm{Q}\omega:=\omega+\alpha\wedge\xi$, so that $\mathrm{D}=\mathrm{d}\circ\mathrm{Q}$. By construction, both D and Q are equivariant with respect to contactomorphisms, i.e. $\mathrm{D}(\phi^*\omega)=\phi^*\mathrm{D}\omega$ and similarly for Q. The Rumin operator fits into a sequence of differential operators defining the so-called Rumin–de Rham complex, whose cohomology is isomorphic to the usual de Rham cohomology [22]. For applications of the Rumin operator in the theory of valuations, see the notes below.

Let us now construct some more operators, which may be seen as refinements of Rumin's construction and which will play an essential role in the following. We use a locally defining 1-form α, and will show later on that the construction is independent of this choice.

Let

$$\mathrm{L}:\Omega^*(M)\to\Omega^{*+2}(M),\ \omega\mapsto\omega\wedge\mathrm{d}\alpha$$

be the Lefschetz operator.

A form $\pi\in\Omega^i(M)$ is called primitive if $i\leq n$ and $\mathrm{L}^{n-i+1}\pi|_Q=0$. Any form $\omega\in\Omega^a(M)$ can be decomposed as

$$\omega|_Q=\sum_{\substack{0\leq i\leq\min(a,2n-a)\\ i\equiv a \mod 2}}\mathrm{L}^{\frac{a-i}{2}}\pi_i|_Q \tag{1}$$

with $\pi_i\in\Omega^i(M)$ primitive, and $\pi_i|_Q$ is unique.

We define for $0\leq i\leq\min(a-2,2n-a), i\equiv a \mod 2$ the maps

$$\mathrm{P}_{a,i}:\Omega^a(M)\to\Omega^{a-1}(M),\ \omega\mapsto\mathrm{L}^{\frac{a-i-2}{2}}\pi_i\wedge\alpha.$$

Then $\mathrm{P}_{a,i}$ is compatible with contactomorphisms. The Rumin operator may be written as

$$\mathrm{D}=\mathrm{d}+(-1)^n\sum_{\substack{0\leq i\leq n-1\\ i\equiv n-1 \mod 2}}\mathrm{d}\circ\mathrm{P}_{n+1,i}\circ\mathrm{d}.$$