Differential Geometry and its Applications 50 (2017) 52–70

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Remarks on symplectic sectional curvature ☆

Daniel J.F. Fox

Departamento de Matemáticas del Área Industrial, Escuela Técnica Superior de Ingeniería y Diseño Industrial, Universidad Politécnica de Madrid, Ronda de Valencia 3, 28012 Madrid, Spain

A R T I C L E I N F O

Article history:
Received 26 August 2015
Received in revised form 19 October 2016
Available online xxxx
Communicated by M.G. Eastwood

MSC:
53D05
53C05

Keywords:
Symplectic connection
Symplectic sectional curvature
Symplectic Lie group

A B S T R A C T

In [11], I.M. Gelfand, V. Retakh, and M. Shubin defined the symplectic sectional curvature of a torsion-free connection preserving a symplectic form. The present article defines the corresponding notion of constant symplectic sectional curvature and characterizes this notion in terms of the curvature tensor of the symplectic connection and its covariant derivatives. Some relations between various more general conditions on the symplectic sectional curvature and the geometry of the symplectic connection or that induced on a symplectic submanifold are explored as well.

1. Introduction

Let (M, Ω) be a connected smooth symplectic manifold of dimension $2n \geq 2$, oriented by the volume form Ω^n. An affine connection on M is *symplectic* if it is torsion free and $\nabla\Omega = 0$. Although the most accessible examples of symplectic connections are Levi-Civita connections of Kähler, pseudo-Kähler, and para-Kähler metrics, even symplectic manifolds that admit no such compatible metric structure admit symplectic connections, for every symplectic manifold admits symplectic connections (see (2.1)). Traditionally, much of the interest in symplectic connections has focused on their use in schemes of deformation quantization such as that of Fedosov [10], but because they exist on any symplectic manifold, it is also interesting to study their geometry in the spirit of classical metric differential geometry.

The basic identities for the curvature of a symplectic connection were probably first obtained by I. Vaisman in [19], although there is information in earlier work of other authors, for example A. Lichnerowicz's [14] and [15]. Among the many available references some basic ones are [1], [10], and [11], in addition to [19]. The survey [4] is a good starting point and contains further references.

☆ Research partially supported by grant MTM2013-42870-P from Spain's Ministerio de Economía y Competitividad.
E-mail address: daniel.fox@upm.es.

http://dx.doi.org/10.1016/j.difgeo.2016.11.001
0926-2245/

The relation between the geometry determined by a symplectic connection ∇ and conditions on quantities and tensors constructed from its curvature is incompletely understood. For example, in [11], I.M. Gelfand, V. Retakh, and M. Shubin defined the symplectic sectional curvature of a symplectic connection. Not much has been done with this notion, and its geometric content has been little explored. This note defines the corresponding notion of constant symplectic sectional curvature, characterizes it in terms of the curvature tensor of the symplectic connection and its covariant derivatives, and describes some related constructions.

The symplectic sectional curvature of a symplectic 2-plane L is a quadratic form on L rather than a number (see Section 4 for the definition), as in the metric setting. Theorem 4.1 shows that the symplectic sectional curvature is determined entirely by the restriction of the Ricci tensor exactly when the symplectic Weyl tensor vanishes. As a consequence, it is sensible to say that a symplectic connection has constant symplectic sectional curvature if for every symplectic 2-plane L it equals the restriction of the quadratic form determined by a *parallel* symmetric two tensor. Corollary 4.1 shows that a symplectic connection has constant symplectic sectional curvature if and only if its symplectic Weyl tensor vanishes and its Ricci tensor is parallel, in which case it is locally symmetric. Corollary 4.2 shows that in this case the Ricci endomorphism (obtained by raising one index of the Ricci tensor using the symplectic form) is parallel, and must be nilpotent of order two, complex, or paracomplex. These conclusions are closely related to and in part can be obtained from those obtained for homogeneous symplectic connections in [7] and for symplectic symmetric spaces (as defined by P. Bieliavsky in [2]) in [8], as is explained in more detail in Remark 3.1. The relation between the symplectic sectional curvatures and the (para-)holomorphic sectional curvatures of a pseudo-Kähler or para-Kähler structure is also discussed. The precise relation is given in Lemma 4.1, that has as a corollary the result of [11] that the symplectic sectional curvatures of a Kähler metric cannot have indefinite signature. Corollary 4.3 shows that ∇ has constant nonzero symplectic sectional curvature if and only if it is a complex projective or complex hyperbolic space form.

The study of pseudo-Riemannian manifolds has in common with the study of symplectic connections that the sectional curvature is defined only for nondegenerate subspaces. A Lorentzian manifold of dimension at least 3 has constant sectional curvature if and only if it has vanishing null sectional curvature (the definitions are recalled in Section 5). In Section 5 there is defined for a symplectic connection a notion of vanishing isotropic sectional curvature analogous to the notion of vanishing null sectional curvature, and Lemma 5.1 shows that on a symplectic manifold of dimension at least 4 a symplectic connection has constant symplectic sectional curvature if and only if it has vanishing isotropic sectional curvature.

In Section 6 there is computed the relation between the symplectic sectional curvature of a symplectic connection and the symplectic sectional curvature of the symplectic connection induced on a symplectic submanifold. The absence of the positivity provided by a Riemannian metric means that the formulas obtained are not as obviously useful as their metric counterparts, for example, in the Kähler setting. There is no obvious analogue of the mean curvature vector field, nor any other tensor associated with a submanifold that is linear in the second fundamental form, other than the form itself. However, the formula (6.8) for the symplectic sectional curvature of a symplectic submanifold helps identify two tensors (defined in (6.9)) constructed from expressions quadratic in the second fundamental form. Precisely, if $\Pi_{ij}{}^{A}$ is the second fundamental form, these tensors are the pure trace part $\mathcal{H}_{ij}$ and the trace-free part $\mathcal{H}_{ijkl}$ of $\Pi_{ik}{}^{A}\Pi_{jl}{}^{B}\Omega_{AB} + \Pi_{il}{}^{A}\Pi_{jk}{}^{B}\Omega_{AB}$ (see Section 6 for the notational conventions used). It would be interesting to understand the geometric meaning of the vanishing of $\mathcal{H}_{ij}$ or $\mathcal{H}_{ijkl}$.

In the metric setting the distance and volume determined by the metric are linked to the curvature via the formulas for their first and second variations. In particular, Jacobi fields are a basic tool. In the symplectic setting it is difficult to link directly the curvature with genuinely geometric quantities, but there is a more serious difficulty that is readily apparent when one tries to mimic metric arguments using Jacobi fields. The symplectic curvature tensor is symmetric rather than antisymmetric in two of its arguments, and this means that quantities that vanish in the metric setting do not vanish in the symplectic setting. This apparently silly technical problem makes it more difficult to control Jacobi fields. What little can be gleaned easily is