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A family of warped product semi-Riemannian Einstein metrics

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ABSTRACT

We study warped product semi-Riemannian Einstein manifolds. We consider the case in that the base is conformal to an n-dimensional pseudo-Euclidean space and invariant under the action of an $(n-1)$-dimensional translation group. We provide all such solutions in the case Ricci-flat when the base is conformal to an n-dimensional pseudo-Euclidean space, invariant under the action of an $(n-1)$-dimensional translation group and the fiber F is Ricci-flat. In particular, we obtain explicit solutions, in the case vacuum, for the Einstein field equation.

1. Introduction and main statements

A semi-Riemannian manifold (M, g) is Einstein if there exists a real constant λ such that

$$Ric_g(X, Y) = \lambda g(X, Y)$$

for each X, Y in T_pM and each p in M. This notion is relevant only for $n \geq 4$. Indeed, if $n = 1$, $Ric_g = 0$. If $n = 2$, then at each p in M, we have $Ric_g(X, Y) = \frac{1}{2}Kg(X, Y)$, so a 2-dimensional semi-Riemannian manifold is Einstein if and only if it has constant sectional or scalar curvature. If $n = 3$, then (M, g) is Einstein if and only if it has constant sectional curvature.

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Over the last few years, several authors have considered the following problem:

Let (M, g) be a semi-Riemannian manifold of dimension $n > 3$. Does there exist a metric g' on M such that (M, g') is an Einstein manifold?

According to T. Aubin [1], to decide if a Riemannian manifold carries an Einstein metric, will be one of the important questions in Riemannian geometry for the next decades. Finding solutions to this problem is equivalent to solving a nonlinear system of second-order partial differential equations. In particular, semi-Riemannian Einstein manifolds with zero Ricci curvature are special solutions, in the vacuum case $(T = 0)$, of the following equation

$$Ric_g - \frac{1}{2}Kg = T, \tag{1}$$

where K the scalar curvature of g and T is a symmetric tensor of order 2. If g is the Lorentz metric on a four-dimensional manifold, this is simply the Einstein field equation. Whenever the tensor T represents a physical field such as electromagnetic field perfect fluid type, pure radiation field and vacuum $(T = 0)$, the above equation has been studied in several papers, most of them dealing with solutions which are invariant under some symmetry group of the equation (see [13] for details). When the metric g is conformal to the Minkowski space–time, then the solutions in the vacuum case are necessarily flat (see [13]).

Several authors constructed new examples of Einstein manifolds. In [15], Ziller constructed examples of compact manifolds with constant Ricci curvature. Chen, in [6], constructed new examples of Einstein manifolds with odd dimension, and, in [16], Yau presented a survey on Ricci-flat manifolds. In [10], Kühnell studied conformal transformations between Einstein spaces and, as a consequence of the obtained results, showed that there is no Riemannian Einstein Manifold with non-constant sectional curvature which is locally conformally flat. This result was extended to semi-Riemannian manifolds (see [12]). Accordingly to construct examples of Einstein manifolds with non-constant sectional curvature, we work with manifolds that are not locally conformally flat. A chance to build these manifolds is to work with warped product manifolds. Then considering (B, g_B) and (F, g_F) semi-Riemannian manifolds, and let $f > 0$ be a smooth function on B, the warped product $M = B \times_f F$ is the product manifold $B \times F$ furnished with the metric tensor

$$\tilde{g} = g_B + f^2 g_F,$$

B is called the base of $M = B \times_f F$, F is the fiber and f is the warping function. For example, polar coordinates determine a warped product in the case of constant curvature spaces, the case corresponds to $\mathbb{R}^+ \times_r S^{n-1}$.

There are several studies correlating warped product manifolds and locally conformally flat manifolds, see [3–5] and their references.

In a series of papers, the authors studied warped product Einstein manifolds under various conditions on the curvature and symmetry, see [7–9]. Particularly, He–Petersen–Wylie, [8], characterized warped product Riemannian Einstein metrics when the base is locally conformally flat.

It is well-known that the Einstein condition on warped geometries requires that the fibers must be necessarily Einstein (see [2,11]). In this paper, initially we give a characterization for warped product semi-Riemannian Einstein manifold when the base is locally conformally flat. Using this characterization, we present new examples of semi-Riemannian manifolds with zero Ricci curvature. More precisely, let us consider $(\mathbb{R}^n, g)$ the pseudo-Euclidean space, $n \geq 3$, with coordinates $x = (x_1, \cdots, x_n)$, $g_{ij} = \delta_{ij}\varepsilon_i$ and $M = (\mathbb{R}^n, \overline{g}) \times_f F^m$ a warped product, where $\overline{g} = \dfrac{1}{\varphi^2}g$, F is a semi-Riemannian Einstein manifold with constant Ricci curvature λ_F, $m \geq 1$, $f, \varphi : \mathbb{R}^n \to \mathbb{R}$ are smooth functions, where f is a positive function. In Theorem 1.1, we find necessary and sufficient conditions for the warped product metric $\widetilde{g} = \overline{g} + f^2 g_F$ to be Einstein. In Theorem 1.2, we consider f and φ invariant under the action of an $(n-1)$-dimensional