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Gauss–Bonnet formulae and rotational integrals in constant curvature spaces

S. Barahona [a], X. Gual-Arnau [b,*]

[a] Departament de Matemàtiques, Universitat Jaume I, 12071-Castelló, Spain
[b] Departament de Matemàtiques, Institute of New Imaging Technologies, Universitat Jaume I, 12071-Castelló, Spain

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ABSTRACT

We obtain generalizations of the main result in [10], and then provide geometric interpretations of linear combinations of the mean curvature integrals that appear in the Gauss–Bonnet formula for hypersurfaces in space forms M_λ^n. Then, we combine these results with classical Morse theory to obtain new rotational integral formulae for the k-th mean curvature integrals of a hypersurface in M_λ^n.

1. Introduction

This paper presents some formulas for the mean curvature integrals of a closed hypersurface in a space of constant curvature λ. The formulas involve the average of certain quantities (measurement functions) evaluated on the intersection of the hypersurface with a random geodesic subspace by a fixed point. Classical results in integral geometry give similar formulas where the random subspaces are not restricted to contain a fixed point. Recently, with a view towards stereological applications (e.g. in microscopy), there has been interest in obtaining such formulas where the subspaces go through a fixed point (see e.g. [1,3,4,11,12]). Most results concern the Euclidean case $\lambda = 0$, but the case $\lambda \neq 0$ is also of interest and has been treated in references [3–6] of the paper. The results in this paper are a natural continuation of those.

Let M_λ^n denote a simply connected Riemannian manifold of constant sectional curvature λ. Further, let L_r^n denote a r-plane ($r \leq n$) namely a totally geodesic submanifold of dimension r in M_λ^n, and let $\mathrm{d}L_r^n$ denote

* Corresponding author.
E-mail addresses: barahona@uji.es (S. Barahona), gual@uji.es (X. Gual-Arnau).

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the corresponding density, invariant under the group of Euclidean and non-Euclidean motions. A r-plane through a fixed point O in M^n_λ, and its invariant density, are denoted by $L^n_{r[0]}$ and $\mathrm{d}L^n_{r[0]}$, respectively [9].

In [4] a new expression for the density of r-planes in M^n_λ has been obtained in terms of the density $\mathrm{d}L^n_{r+1[0]}$, of the density $\mathrm{d}L^{r+1}_r$ of r-planes in $L^n_{r+1[0]}$ and the distance ρ from O to L^{r+1}_r. Thus, an invariant r-plane in M^n_λ may be generated by taking first an isotropic $(r+1)$-plane through a fixed point O and then an invariant r-plane within this $(r+1)$-plane, weighted by a function of ρ.

This construction, called the invariator principle in M^n_λ ([11]), has opened the way to solve rotational integral equations for different quantities as the volume of a k-dimensional submanifold in M^n_λ [4], the k-th mean curvature integrals or k-th intrinsic volumes ([6] and [1], and different curvature measures ([11] for $\lambda = 0$)). The solutions of these equations allow to express these quantities as the integral of some functionals defined in sections produced by isotropic planes through a fixed point. Moreover, in [11], the authors, using classical Morse theory, rewrite the volume of compact submanifolds in $\mathbb{R}^n$ of dimension $n-r$, in terms of critical values of the sectioned object with $(r+1)$-planes; and in [5] related generalizations valid for submanifolds in space forms of constant curvature are obtained.

If we compare some classical formulas in integral geometry, obtained by sections which do not necessarily contain a fixed point of reference, with rotational formulas in spaces of constant curvature; we obtain the following equivalences. In [4] rotational formulae of Eq. (14.69) of [9] are obtained. In [5] we give a Morse type representation of these formulas. In [6] rotational formulae of Eq. (14.78) of [9] are obtained; then, the mean curvature integrals of the sectioned domain appear in the measurement functions. In [3], we give stereological versions in $\mathbb{R}^3$ of all the preceding integral formulae. In this paper we give rotational formulae for the mean curvature integrals, from the Gauss–Bonnet formula in non-Euclidean spaces (Eq. (17.21) and Eq. (17.22) of [9]) and Eq. (17) of this paper; therefore, the Euler characteristic of the sectioned domain appears in the measurement functions, and we give a Morse type representation of these formulas.

On the other hand, in [10] it is proved that the Gauss–Bonnet defect of a hypersurface in M^n_λ is the measure of planes L^n_{n-2} meeting it, counted with multiplicity. From this result an integral-geometric proof of the Gauss–Bonnet theorem for hypersurfaces in M^n_λ is given.

The purpose of this paper is twofold: to obtain generalizations of the main result in [10], following a completely different route; and to combine these results with classical Morse theory to obtain new rotational integral formulae for the k-th mean curvature integrals of a hypersurface in M^n_λ.

2. The Gauss–Bonnet theorem in M^n_λ

Let $Q \subset M^n_\lambda$ be a compact domain with smooth boundary $S = \partial Q$. Let V denote the volume of Q, F the $(n-1)$-surface area of S, $\chi(Q)$ the Euler–Poincaré characteristic of Q, and M_i the i-th integral of mean curvature of S. The Gauss–Bonnet formula for S states that [9]

$$c_{n-1}M_{n-1} + \lambda c_{n-3}M_{n-3} + \cdots + \lambda^{\frac{n-2}{2}} c_1 M_1 + \lambda^{\frac{n}{2}} V = \frac{1}{2} O_n \chi(Q), \tag{1}$$

for n even, where $O_k = \mathrm{vol}(\mathbb{S}^k)$ (surface area of the k-dimensional unit sphere), and

$$c_{n-1}M_{n-1} + \lambda c_{n-3}M_{n-3} + \cdots + \lambda^{\frac{n-3}{2}} c_2 M_2 + \lambda^{\frac{n-1}{2}} c_0 F = \frac{1}{2} O_n \chi(Q), \tag{2}$$

for n odd, where

$$c_h = \binom{n-1}{h} \frac{O_n}{O_h O_{n-1-h}}. \tag{3}$$

If n is odd, we can use the equality $2\chi(Q) = \chi(S)$, and for $\lambda = 0$, in any case, we obtain $M_{n-1} = O_{n-1}\chi(Q)$.