Differential Geometry and its Applications 50 (2017) 155–166

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Biharmonic maps from biconformal transformations with respect to isoparametric functions ☆

P. Baird *, A. Fardoun, S. Ouakkas

ARTICLE INFO

Article history:
Received 16 January 2016
Available online 8 December 2016
Communicated by V. Souček

MSC:
53A30
31B30
53A10

ABSTRACT

By exploiting biconformal transformations of the metric we construct biharmonic functions and mappings from Riemannian manifolds. Isoparametric functions, characterized by the property that their level sets are parallel and of constant mean curvature, play an important role in the construction of examples. We extend our method to include triconformal deformations of the metrics with respect to the Hopf map from $\mathbb{R}^4$ to $\mathbb{R}^3$, which, in addition to a biconformal deformation of the domain, incorporates a conformal deformation of the codomain.

1. Introduction

A biharmonic map is a mapping $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds which is critical for the bienergy functional (see below):

$$\int_{M^m} |\tau_\varphi|^2 dv_g\,,$$

where τ_φ is the tension field of φ. The origins of this functional can be traced back to Euler's elastica, important in elasticity theory, which arise as extrema of the total squared curvature functional for a curve with the constraint that its length be fixed [10]. More generally the functional without constraint was introduced for curves in arbitrary Riemannian manifolds by L. Noakes, G. Heinzinger and B. Paden; in this case extrema are known as *Riemannian cubics* [14]. Independently, biharmonic curves in surfaces and in the Heisenberg group have been investigated by R. Caddeo, S. Montaldo and P. Piu [6] and R. Caddeo, C. Oniciuc and P. Piu [7], respectively. In recent years, biharmonic maps have been studied from more

☆ For part of this work, the first author was supported by a 2015 Short Stay Fellowship Grant awarded by the Institute of Advanced Studies, as well as by the School of Mathematics and Engineering Science at the University of Western Australia. He would like to thank Lyle Noakes for drawing his attention to Euler's elastica and to the theory of Riemannian cubics.

* Corresponding author.
E-mail address: Paul.Baird@univ-brest.fr (P. Baird).

http://dx.doi.org/10.1016/j.difgeo.2016.12.001
0926-2245/

general domains and we refer the reader to [12] for a short survey. The corresponding Euler–Lagrange equations are a 4th order elliptic system. For curves in the plane these can be integrated and, as made explicit by Euler, their solutions can be expressed in terms of elliptic functions. For more general domains and codomains the Euler–Lagrange equations are difficult to deal with.

By definition, a harmonic map has $\tau_\varphi \equiv 0$ which is therefore automatically biharmonic; so one is interested in finding biharmonic maps which are non-harmonic, so-called *proper* biharmonic maps. One approach is to fix a map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds and to deform the metric g on the domain or the metric h on the codomain in order to render the map biharmonic. This idea was first considered in [2], where it was shown that if $\widetilde{g} = e^{2\gamma} g$ is a conformally related metric ($\gamma : M^m \to \mathbb{R}$ a smooth function) and φ is *harmonic*, then the deformed metric $\widetilde{g}$ renders φ biharmonic if and only if the gradient $\nabla\gamma$ satisfies a 2nd order partial differential equation. Thus the 4th order system is now equivalent to two second order systems. In general these are still difficult to solve, but by taking φ to be particularly simple, for example the identity map, one can construct proper biharmonic maps.

Conformal deformations were also considered by the third author [15,16] who characterized conformal biharmonic maps in terms of partial differential equations, once more exploiting conformal deformations of the codomain. Equations characterizing semi-conformal biharmonic maps were obtained by the authors in [3].

In the case that φ is a submersion, then at each point $x \in M^m$, the tangent space has an orthogonal decomposition $T_x M^m = \mathcal{H}_x \oplus \mathcal{V}_x$ as a direct sum of its vertical (tangent to the fibres of φ) and horizontal spaces. The metric g also decomposes into a sum $g = g^{\mathcal{H}} + g^{\mathcal{V}}$, where $g^{\mathcal{V}}$ is defined by $g^{\mathcal{V}}|_{\mathcal{V}} = g|_{\mathcal{V}}$ and $g^{\mathcal{V}}|_{\mathcal{H}} = 0$. A *biconformal deformation of* g is a metric of the form:

$$\widetilde{g} = \sigma^{-2} g^{\mathcal{H}} + \rho^{-2} g^{\mathcal{V}}$$

where σ and ρ are smooth positive functions on M^n. If in addition we deform the metric on the codomain to $\widetilde{h} = \nu^{-2} h$, we call the deformation *triconformal*. Such a deformation preserves the orthogonal decomposition of TM^m into horizontal and vertical subspaces, but dilates these spaces by different amounts. If $\sigma \equiv \rho$, then a biconformal deformation is conformal.

In his thesis, L. Danielo used biconformal deformations to construct new examples of Einstein metrics in dimension 4 [8]. The deformations exploited were with respect to standard projections and the Hopf maps. In this article, we show how to construct proper biharmonic maps from both biconformal and triconformal deformations. For a function $\varphi : M^m \to \mathbb{R}$, the biharmonic map equations are simply the square of the Laplace–Beltrami operator: $(\Delta)^2\varphi = \Delta(\Delta\varphi) = 0$. In the first instance, we exploit isoparametric functions, characterized by the property that their level sets are parallel and of constant mean curvature, in order to construct biharmonic functions. A connection with isoparametric functions was also found in [3], where it was shown that for dimension $m \neq 4$, any biharmonic conformal mapping of M^m has dilation which is an isoparametric function.

In Theorem 4, we give a natural generalization of this approach to semi-conformal maps between Riemannian manifolds with basic tension field and basic dilation. By considering projections we give examples of proper biharmonic maps on complete non-compact Riemannian manifolds. We are particularly interested in entire examples because of the Liouville type-theorems for biharmonic maps obtained in the references [4,13].

By an application of the Weitzenböck formula, it was shown by G.Y. Jiang that any biharmonic map from a compact manifold without boundary into a manifold of non-positive sectional curvature must be harmonic [11]. However, there is no evident reason why there should not exist a proper biharmonic submersion between compact manifolds of positive curvature. In spite of this, there are no known examples when the dimension of the domain is ≥ 2. One strategy might be to begin with say the Hopf map from $\mathbb{R}^4$ to $\mathbb{R}^3$ and to conformally deform the metric on the codomain into a compact metric such as a spherical one. A semi-conformal *harmonic* map from a deformed S^4 into Euclidean S^3 was constructed in this way