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Bernoulli–Dunkl and Apostol–Euler–Dunkl polynomials with applications to series involving zeros of Bessel functions☆

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Abstract

We introduce Bernoulli–Dunkl and Apostol–Euler–Dunkl polynomials as generalizations of Bernoulli and Apostol–Euler polynomials, where the role of the derivative is now played by the Dunkl operator on the real line. We use them to find the sum of many different series involving the zeros of Bessel functions.

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1. Introduction

Along the middle years of the XVIIIth century, Euler proved that

$$\sum_{j=1}^{\infty} \frac{1}{j^{2k}} = \frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!} B_{2k}, \quad k = 1, 2, \dots, \tag{1.1}$$

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{(2j-1)^{2k+1}} = \frac{(-1)^{k+1}\pi^{2k+1}}{4(2k)!} E_{2k}, \quad k = 0, 1, 2, \dots, \tag{1.2}$$

where B_{2k} and E_{2k} are the Bernoulli and Euler numbers, respectively. Bernoulli and Euler numbers are the particular values $B_{2k} = B_{2k}(0)$ and $E_{2k} = E_{2k}(1/2)$, where $\{B_n(x)\}_n$ and $\{E_n(x)\}_n$ are, respectively, the Bernoulli and Euler polynomials defined by the generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \qquad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

The sum (1.1) can be seen as an identity for the sum of the reciprocals of the zeros of $\sin(x)$ while in the sum (1.2) the zeros of $\cos(x)$ are modified by an alternating sign. The sine and cosine functions can be expressed in terms of Bessel functions: $\sin(x) = (\pi x/2)^{1/2} J_{1/2}(x)$ and $\cos(x) = (\pi x/2)^{1/2} J_{-1/2}(x)$, respectively. So a natural generalization of (1.1) is to use Bessel functions $J_\alpha(x)$ (see, for instance, [26,17,19]) and to compute the series

$$\sum_j \frac{1}{s_{j,\alpha}^{2k}}, \tag{1.3}$$

where $\{s_{j,\alpha}\}_j$ are the zeros of a Bessel function. This question, which has both mathematical and physical interest, has a somehow classical flavour and goes back to Lord Rayleigh in 1874 [21] (in fact, the sum (1.3) as a function of α is usually called the Rayleigh function). Since then, many papers have been published which study, with different approaches, these and many other series along with identities and other properties; see, for instance, [24,16,15,3,10] or [20, formula 11 in § 5.7.33] (and the list is by no means exhaustive).

However, identities relating (1.3) or any other sum involving the zeros of Bessel functions with some kind of "Bernoulli" or "Euler numbers" seem to be unknown. Such identities could be considered a true generalization of (1.1) and (1.2).

The purpose of this paper is to introduce what we have called Bernoulli–Dunkl and Apostol–Euler–Dunkl polynomials. We will use them to sum many series involving the zeros of Bessel functions, among which are the analogous to the series (1.1) and (1.2).

Our approach is the following. Bernoulli and Euler polynomials are particular cases of the so-called Appell sequences. An Appell sequence $\{P_n(x)\}_{n=0}^{\infty}$ is a sequence of polynomials defined by a Taylor generating expansion

$$A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}, \tag{1.4}$$

where $A(t)$ is a function analytic at $t = 0$ with $A(0) \neq 0$. Since the exponential function e^x is invariant under the differential operator d/dx, it is easy to show that $P_n(x)$ is a polynomial of degree n and $P_n'(x) = nP_{n-1}(x)$. Typical examples of Appell sequences are the Bernoulli and Euler polynomials above, or the probabilistic Hermite polynomials $\{\mathrm{He}_n(x)\}_{n=0}^{\infty}$.