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Estimates for Entropy Numbers of Sets of Smooth Functions on the Torus $\mathbb{T}^d$

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Abstract

In this paper, we investigate entropy numbers of multiplier operators $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ and $\Lambda_* = \{\lambda^*_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\Lambda, \Lambda_* : L^p(\mathbb{T}^d) \to L^q(\mathbb{T}^d)$ on the d-dimensional torus $\mathbb{T}^d$, where $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$ and $\lambda^*_{\mathbf{k}} = \lambda(|\mathbf{k}|_*)$ for a function λ defined on the interval $[0,\infty)$, with $|\mathbf{k}| = \left(k_1^2 + \cdots + k_d^2\right)^{1/2}$ and $|\mathbf{k}|_* = \max_{1\le j\le d} |k_j|$. In the first part, upper and lower bounds are established for entropy numbers of general multiplier operators. In the second part, we apply these results to the specific multiplier operators $\Lambda^{(1)} = \left\{|\mathbf{k}|^{-\gamma}(\ln|\mathbf{k}|)^{-\xi}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\Lambda^{(1)}_* = \left\{|\mathbf{k}|_*^{-\gamma}(\ln|\mathbf{k}|_*)^{-\xi}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\Lambda^{(2)} = \left\{e^{-\gamma|\mathbf{k}|^r}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$ and $\Lambda^{(2)}_* = \left\{e^{-\gamma|\mathbf{k}|_*^r}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$ for $\gamma > 0$, $0 < r \le 1$ and $\xi \ge 0$. We have that $\Lambda^{(1)}U_p$ and $\Lambda^{(1)}_*U_p$ are sets of finitely differentiable functions on $\mathbb{T}^d$, in particular, $\Lambda^{(1)}U_p$ and $\Lambda^{(1)}_*U_p$ are Sobolev-type classes if $\xi = 0$, and $\Lambda^{(2)}U_p$ and $\Lambda^{(2)}_*U_p$ are sets of infinitely differentiable ($0 < r < 1$) or analytic ($r = 1$) functions on $\mathbb{T}^d$, where U_p denotes the closed unit ball of $L^p(\mathbb{T}^d)$. In particular, we prove that, the estimates for the entropy numbers $e_n(\Lambda^{(1)}U_p, L^q(\mathbb{T}^d))$, $e_n(\Lambda^{(1)}_*U_p, L^q(\mathbb{T}^d))$, $e_n(\Lambda^{(2)}U_p, L^q(\mathbb{T}^d))$ and $e_n(\Lambda^{(2)}_*U_p, L^q(\mathbb{T}^d))$ are order sharp in various important situations.

Keywords: Torus, Entropy, Multiplier, Smooth function.

1. Introduction

In [11], [13], [14] techniques were developed to obtain estimates for entropy numbers of multiplier operators defined for functions on the sphere $\mathbb{S}^d$ and on two-points homogeneous spaces. In this paper, we obtain estimates for entropy numbers of multiplier operators $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ and $\Lambda_* = \{\lambda^*_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ defined for functions on the d-dimensional torus $\mathbb{T}^d$, where $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$ and $\lambda^*_{\mathbf{k}} = \lambda(|\mathbf{k}|_*)$, for a function λ defined on $[0,\infty)$, with $|\mathbf{k}| = \left(k_1^2 + \cdots + k_d^2\right)^{1/2}$ and $|\mathbf{k}|_* = \max_{1\le j\le d} |k_j|$.

The entropy numbers measure a kind of degree of compactness of an operator and have many applications in theory of functions and spectral theory ([8] and [18]), signals and image processing ([6] and [7]), probability theory ([6]), among others.

Estimates for entropy numbers of Sobolev classes on the torus were also obtained in [1]. Studies about approximation for functions in Sobolev classes on the torus were made recently in [4] and [12]. For the interested reader, two important monographs on entropy numbers are [5] and [3].

In the first part of this paper we give an unified treatment for entropy numbers of sets of functions determined by multiplier operators. Upper and lower bounds are established for entropy numbers of general multiplier operators. Among the tools used in the proofs of these results, the main is a theorem proved in [10], which provides estimates for Levy means of norms defined on $\mathbb{R}^n$. In the second part, we apply these results in the study of estimates for entropy numbers of sets of finitely and infinitely differentiable functions on $\mathbb{T}^d$. We show, in particular, that in various important situations the estimates are order sharp. An important tool used in the second part is the estimate for the

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