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# Nested sequences of rational spaces: Bernstein approximation, dimension elevation, and Pólya-type theorems on positive polynomials

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**Abstract**

On a given closed bounded interval, an infinite nested sequence of Extended Chebyshev spaces containing the constants automatically generates an infinite sequence of positive linear operators of Bernstein-type. Unlike the polynomial framework, this situation does not guarantee convergence of the corresponding approximation process. Obviously, convergence cannot be obtained without the density of the union of all the involved spaces in the set of continuous functions equipped with the uniform norm. The initial purpose of this work was to answer the following question: conversely, is density sufficient to guarantee convergence? Addressing this issue is all the more natural as density was indeed proved to imply convergence in the special case of nested sequences of Müntz spaces on positive intervals. In this paper we give a negative answer to the aforementioned question by considering nested sequences of rational spaces defined by infinite sequences of real poles outside the given interval. Surprisingly, in this rational context, we show that ensuring convergence is equivalent to determining all Pólya positive sequences, in the sense of all infinite sequences of positive numbers which guarantee Pólya-type results for the positivity of univariate polynomials on the non-negative axis. This interesting connection with Pólya positive sequences enables us to produce a simple necessary and sufficient condition for the poles to ensure convergence, thanks to results by Baker and Handelman on strongly positive sequences of polynomials.

**Keywords:** Approximation by rational functions - Bernstein-type operators - Extended Chebyshev spaces - Dimension elevation - Positive polynomials - Pólya-type theorems

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## 1. Introduction

Throughout the article, $[a,b]$, $a < b$, is a fixed interval. For any non-negative integer $n$, we denote by $\mathbb{P}_n$ the set of all polynomial functions on $[a,b]$ of degree at most $n$. Two

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