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Full Length Article

# Optimal recovery of three times differentiable functions on a convex polytope inscribed in a sphere

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**Abstract**

We consider the problem of global recovery on the class $W^3(P)$ of three times differentiable functions which have uniformly bounded third order derivatives in any direction on a $d$-dimensional convex polytope $P$ inscribed in a sphere and containing its circumcenter. The information $I(f)$ known about each function $f \in W^3(P)$ is given by its values and gradients at the vertices of $P$. The recovery error is measured in the uniform norm on $P$. We prove the optimality on the class $W^3(P)$ of a certain quasi-interpolating recovery method among all non-adaptive global recovery methods which use the information $I(f)$. This method was constructed earlier for the case of a $d$-dimensional simplex $T$ in the work by the author and T.S. Sorokina in 2011, where its optimality was proved for an analogous class $W^2(T)$ of twice differentiable functions.


*Keywords:* Multivariate spline; Optimal recovery; Worst-case setting; Sobolev class

## 1. Rigorous setting of the problem

The worst-case error setting of the optimal recovery problem considered in this paper goes back to the works by Kolmogorov from the 1940s. First results on this problem were obtained by Sard [24], Nikol'skii [22], and Kiefer [11]. Later, a large number of optimal recovery results were obtained by many authors. Detailed reviews of these results can be found in the books [27,13,26,14,21,28,23].

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Throughout the paper, let $P = P(\mathbf{v}_1, \dots, \mathbf{v}_m)$ be a convex polytope in $\mathbf{R}^d$, $d \in \mathbf{N}$, with non-empty interior and the set of vertices $\mathcal{V}_m := \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$; that is, $P$ is the convex hull of the set $\mathcal{V}_m$, every point $\mathbf{v}_i \in \mathcal{V}_m$ is not a convex combination of points $\mathbf{v}_j$, $j \neq i$, and the points from $\mathcal{V}_m$ are in general position (not contained in any $(d-1)$-dimensional hyperplane).

Let $\mathcal{F}$ be a class of continuously differentiable[1] functions defined on the polytope $P$. Assume that for every function $f \in \mathcal{F}$, the following vector

$$I(f) := (f(\mathbf{v}_1), \dots, f(\mathbf{v}_m), \nabla f(\mathbf{v}_1), \dots, \nabla f(\mathbf{v}_m)) \in \mathbf{R}^{m(d+1)} \tag{1}$$

is known. Let $B(P)$ denote the space of bounded functions defined on the polytope $P$. Every mapping $\Phi : \mathbf{R}^{m(d+1)} \to B(P)$ generates an algorithm of global recovery of a function $f$ (based on information (1)) of the form

$$S_f := \Phi[I(f)]. \tag{2}$$

The input of algorithm (2) is a vector of values in (1) sampled from $f$ and the output is a bounded function $S_f$ defined on $P$, which is used as an approximation to the function $f$. Though the function $f$ is differentiable, we still allow output functions $S_f$ that are only bounded since the techniques used do not require differentiability or even continuity of the output function. This enables us to include a wider class of algorithms into our considerations. In our results, the class $\mathcal{F}$ has smoothness higher than one, however, in the general setting of the problem presented below we assume functions from $\mathcal{F}$ to be only continuously differentiable since this is sufficient for the data vector $I(f)$ to be well-defined on the class $\mathcal{F}$.

The (worst-case) error of algorithm (2) over the class $\mathcal{F}$ is given by

$$e(\mathcal{F}; \Phi) := \sup_{f \in \mathcal{F}} \|f - \Phi[I(f)]\|_P, \tag{3}$$

where $\|\cdot\|_P$ is the uniform norm on the polytope $P$; that is,

$$\|u\|_P := \sup_{\mathbf{x} \in P} |u(\mathbf{x})|, \quad u \in B(P).$$

**Problem 1.1.** Find the quantity

$$e(\mathcal{F}; \mathcal{V}_m) := \inf_{\Phi : \mathbf{R}^{m(d+1)} \to B(P)} e(\mathcal{F}; \Phi) \tag{4}$$

and optimal global recovery algorithm(s) that use data given by (1); that is, mapping(s) $\Phi^* : \mathbf{R}^{m(d+1)} \to B(P)$ that attain the infimum on the right-hand side of (4).

Let $W^r_\infty[a, b]$, $r = 1, 2, 3$, be the class of univariate functions $g : [a, b] \to \mathbf{R}$ such that the derivative $g^{(r-1)}$ is absolutely continuous on $[a, b]$ and $\left|g^{(r)}(t)\right| \leq 1$ for almost all $t \in [a, b]$.

**Definition 1.2.** Let $W^r(P)$, $r = 1, 2, 3$, be the class of functions $f$ which are continuous on $P$ in the case $r = 1$, continuously differentiable on $P$ in the case $r = 2$, and twice continuously differentiable in an open set containing $P$ in the case $r = 3$, and are such that for any two

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[1] Recall that $f : P \to \mathbf{R}$ is called differentiable at a point $\mathbf{v} \in P$ if there is a vector $\mathbf{a_v} \in \mathbf{R}^d$ such that

$$f(\mathbf{x}) - f(\mathbf{v}) = \mathbf{a_v} \cdot (\mathbf{x} - \mathbf{v}) + o(\mathbf{x} - \mathbf{v}), \quad \mathbf{x} \to \mathbf{v}, \quad \mathbf{x} \in P.$$

Even though partial derivatives of $f$ at some boundary points $\mathbf{v} \in P$ may be undefined, the vector $\mathbf{a_v}$ is uniquely defined whenever $f$ is differentiable at $\mathbf{v}$. Therefore, we can still speak about the gradient of $f$ by letting $\nabla f(\mathbf{v}) = \mathbf{a_v}$ for all $\mathbf{v} \in P$ and call $f$ continuously differentiable on $P$ if the vector $\mathbf{a_v}$ depends continuously on $\mathbf{v} \in P$.