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Full Length Article

The algebras of difference operators associated to Krall–Charlier orthogonal polynomials☆

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Abstract

Krall–Charlier polynomials $(c_n^{a;F})_n$ are orthogonal polynomials which are also eigenfunctions of a higher order difference operator. They are defined from a parameter a (associated to the Charlier polynomials) and a finite set F of positive integers. We study the algebra $\mathfrak{D}_a^F$ formed by all difference operators with respect to which the family of Krall–Charlier polynomials $(c_n^{a;F})_n$ are eigenfunctions. Each operator $D \in \mathfrak{D}_a^F$ is characterized by the so called eigenvalue polynomial λ_D: λ_D is the polynomial satisfying $D(c_n^{a;F}) = \lambda_D(n)c_n^{a;F}$. We characterize the algebra of difference operators $\mathfrak{D}_a^F$ by means of the algebra of polynomials $\tilde{\mathfrak{D}}_a^F = \{\lambda \in \mathbb{C}[x] : \lambda(x) = \lambda_D(x), D \in \mathfrak{D}_a^F\}$. We associate to the family $(c_n^{a;F})_n$ a polynomial Ω_F^a and prove that, except for degenerate cases, the algebra $\tilde{\mathfrak{D}}_a^F$ is formed by all polynomials $\lambda(x)$ such that Ω_F^a divides $\lambda(x) - \lambda(x-1)$. We prove that this is always the case for a segment F (i.e., the elements of F are consecutive positive integers), and conjecture that it is also the case when the Krall–Charlier polynomials $(c_n^{a;F})_n$ are orthogonal with respect to a positive measure.

MSC: 42C05; 33C45; 33E30

Keywords: Orthogonal polynomials; Krall orthogonal polynomials; Algebra of difference operators; Charlier polynomials

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1. Introduction

Krall polynomials are orthogonal polynomials which are also eigenfunctions of a higher order differential operator (in other words, differential operators of arbitrary order). This issue was raised by H.L. Krall in 1939, when he obtained a complete classification for the case of differential operators of order four [34]. The terminology has been extended for finite order difference and q-difference operators. Hence Krall discrete polynomials are orthogonal polynomials which are also eigenfunctions of higher order difference operators. In the terminology introduced by Duistermaat and Grünbaum ([9]; see also [22,23]), Krall, Krall discrete or q-Krall polynomials are also called bispectral. Krall and Krall discrete polynomials are one of the most important generalizations of the classical (Hermite, Laguerre and Jacobi) and classical discrete (Charlier, Meixner, Krawtchouk and Hahn) families of orthogonal polynomials, respectively.

Since the eighties a lot of effort has been devoted to find Krall polynomials. Roughly speaking, one can construct Krall polynomials $q_n(x)$, $n \geq 0$, by using the Laguerre $x^\alpha e^{-x}$, or Jacobi weights $(1-x)^\alpha(1+x)^\beta$, assuming that one or two of the parameters α and β are nonnegative integers and adding a linear combination of Dirac deltas and their derivatives at the endpoints of the orthogonality interval [34,30,32,31,36,37,22,24–27,42]. Specially relevant for this paper is [27], where P. Iliev characterized the algebra of differential operators with respect to which the Krall–Laguerre polynomials are eigenfunctions.

The procedure of adding deltas seems not to work if we want to construct Krall discrete polynomials from the classical discrete measures of Charlier, Meixner, Krawtchouk and Hahn (see the papers [3,4] by Bavinck, van Haeringen and Koekoek answering, in the negative, a question posed by R. Askey in 1991 (see page 418 of [5])).

The first examples of Krall discrete polynomials have been introduced by the author in [11]. Orthogonalizing measures for these families of polynomials are generated by multiplying the classical discrete weights by certain variants of the annihilator polynomial of a set of numbers. For a finite set of numbers F, this annihilator polynomial $\mathfrak{p}_F$ is defined by

$$\mathfrak{p}_F(x) = \prod_{f\in F}(x-f).$$

The kind of transformation which consists in multiplying a measure $d\mu$ by a polynomial r is called a Christoffel transform. It has a long tradition in the context of orthogonal polynomials: it goes back a century and a half ago when E.B. Christoffel (see [7] and also [40]) studied it for the particular case $\mu(x) = x$.

Using this annihilator polynomial (and other of its variants), a number of conjectures were posed in [11]. All of them have been proved in a sequence of papers [12,17,18], see also [16,19,20,2].

We denote by $\mathcal{A}$ the algebra formed by all higher order difference operators of the form

$$D = \sum_{l=s}^{r} h_l \mathfrak{s}_l, \quad s \leq r, s, r \in \mathbb{Z}, \tag{1.1}$$

where h_l are polynomials and $\mathfrak{s}_l$ stands for the shift operator $\mathfrak{s}_l(p) = p(x+l)$. If $h_r, h_s \neq 0$, the order of D is then $r-s$. We also say that D has genre (s, r).

The simplest case of Krall discrete polynomials is that constructed from the Charlier family. Let μ_a be the Charlier weight

$$\mu_a = \sum_{n=0}^{\infty} \frac{a^n}{n!}\delta_n,$$