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Full Length Article

An optimal approximation formula for functions with singularities

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Abstract

We propose an optimal approximation formula for analytic functions that are defined on a complex region containing the real interval $(-1, 1)$ and possibly have algebraic singularities at the endpoints of the interval. As a space of such functions, we consider a Hardy space with the weight given by $w_\mu(z) = (1 - z^2)^{\mu/2}$ for $\mu > 0$, and formulate the optimality of an approximation formula for the functions in the space. Then, we propose an optimal approximation formula for the space for any $\mu > 0$, whereas μ is restricted as $0 < \mu < \mu_*$ for a certain constant μ_* in the existing result. We also provide the results of numerical experiments to show the performance of the proposed formula.

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Keywords: Hardy space; Endpoint singularity; Optimal approximation; Ganelius sampling points

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1. Introduction

This paper is concerned with approximation of functions by a finite number of the sampled values of them. We consider analytic functions that are defined on a complex region containing a real interval and possibly have endpoint singularities on the interval. In order to deal with such functions collectively, we consider a function space consisting of them and formulate the optimality of an approximation formula for the functions in the space. Then, we propose an optimal approximation formula for the function space.

We consider the region given by

$$\Lambda_d = \left\{ z \in \mathbf{C} \,\middle|\, \left| \arg\left(\frac{1+z}{1-z}\right) \right| < d \right\},$$

which satisfies $\Lambda_d \cap \mathbf{R} = (-1, 1)$. In order to deal with analytic functions on Λ_d with algebraic singularities at the endpoints ± 1, we consider the function space given by

$$\boldsymbol{H}^\infty(\Lambda_d, w_\mu) = \left\{ f : \Lambda_d \to \mathbf{C} \,\middle|\, f \text{ is analytic in } \Lambda_d \text{ and } \sup_{z \in \Lambda_d} \left| \frac{f(z)}{w_\mu(z)} \right| < \infty \right\},$$

where μ is a positive number and $w_\mu(z) = (1 - z^2)^{\mu/2}$. This space has been studied as a fundamental space for the sinc numerical methods [6,8], which are the numerical methods based on the approximation of functions by the sinc function (see (2.6)). The error analysis of the sinc approximation has been performed in these decades [6–8,10]. It is well-known that the sinc approximation has very good accuracy in $\boldsymbol{H}^\infty(\Lambda_d, w_\mu)$.

Besides the studies of such concrete formulas in $\boldsymbol{H}^\infty(\Lambda_d, w_\mu)$, there are several analyses of the errors of optimal formulas in spaces of analytic functions like $\boldsymbol{H}^\infty(\Lambda_d, w_\mu)$. In the papers [1,5,9,12], the authors have estimated the optimal errors in Hardy spaces with preassigned decay rates. In particular, Sugihara [9] has given a lower bound of the optimal error in $\boldsymbol{H}^\infty(\Lambda_d, w_\mu)$ and revealed that the sinc approximation is near optimal in the space. In order to formulate the optimality of an approximation formula for the functions in $\boldsymbol{H}^\infty(\Lambda_d, w_\mu)$, he considered all the possible n-point approximation formulas in the space and the norms of their error operators. Then, he defined the minimum error norm $E_n^{\min}(\boldsymbol{H}^\infty(\Lambda_d, w_\mu))$ by the minimum of the norms. Furthermore, he also considered the error norm of the sinc approximation on $\boldsymbol{H}^\infty(\Lambda_d, w_\mu)$, denoted by $E_n^{\mathrm{sinc}}(\boldsymbol{H}^\infty(\Lambda_d, w_\mu))$, and has shown that

$$c'' \exp(-c_2\sqrt{n}) \leq E_n^{\min}(\boldsymbol{H}^\infty(\Lambda_d, w_\mu)) \leq E_n^{\mathrm{sinc}}(\boldsymbol{H}^\infty(\Lambda_d, w_\mu)) \leq c'\sqrt{n}\exp(-c_1\sqrt{n}),$$

where c', c'', c_1, and c_2 are positive constants with $c_1 < c_2$ (see (2.9)).

However, finding an explicit approximation formula attaining $E_n^{\min}(\boldsymbol{H}^\infty(\Lambda_d, w_\mu))$ has been an open problem so far whereas the exact order of $E_n^{\min}(\boldsymbol{H}^\infty(\Lambda_d, w_\mu))$ with respect to n is known in some restricted case. Recently, in the restricted case that $0 < \mu < \min\{2, \pi/d\}$, Ushima et al. [11] have proposed an optimal formula by using the technique of Jang and Haber [3], in which they employ a modification of the sampling points given by Ganelius [2]. The restriction $0 < \mu < \min\{2, \pi/d\}$ is owing to the assumption $r < 1$ in the Ganelius theorem [3, Lemma 1], which plays an important role for the error estimate of the formula. In this paper, we remove this restriction and propose an optimal formula for any $\mu > 0$ by generalizing the formula in [11].

The rest of this paper is organized as follows. In Section 2, we list mathematical tools for setting the framework for approximation of the functions in $\boldsymbol{H}^\infty(\Lambda_d, w_\mu)$. We give the more precise explanations of the region Λ_d, space $\boldsymbol{H}^\infty(\Lambda_d, w_\mu)$, and the notion of the optimal approximation in $\boldsymbol{H}^\infty(\Lambda_d, w_\mu)$. Furthermore, we review some existing results about the estimate