Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 233 (2018) 58–71

JOURNAL OF Approximation Theory

www.elsevier.com/locate/jat

Full Length Article

Problems and conjectures in approximation that I would like settled

Z. Ditzian

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Received 12 September 2017; received in revised form 8 February 2018; accepted 11 April 2018
Available online 25 April 2018

Communicated by Vilmos Totik

Abstract

Open problems relating smoothness to the rate of approximation that I would like solved are presented with some background. Conjectures for the solutions of some of these problems are given with intuitive reasoning.

MSC: 26A16; 41A10; 41A17; 33C47; 40G05
Keywords: Measure of smoothness; Direct and weak converse inequalities; Cesàro summability

1. Introduction

In this article I present questions and conjectures that occurred to me during my work. Some, but not all, of these will prove to be quite difficult. The context of the questions will be given. That is, what is known and where it was proved, what I believe is valid in the remaining cases (and why) and what applications I hope these new results will have. The work is related to approximation on various domains. The domains emphasized will be the sphere, the circle $[-\pi, \pi)$, the torus, ball, polytopes, R, R^d and other convex sets. The weights treated include Jacobi-type weights on the simplex, $(1 - \|\boldsymbol{x}\|^2)^{\lambda - \frac{1}{2}}$ on the unit ball, Freud-type weights on R, and $w(\boldsymbol{x}) = 1$ on other domains. The emphasis will be on the metric L_p, $p > 0$.

E-mail address: zditzian@shaw.ca.

https://doi.org/10.1016/j.jat.2018.04.001

The Cesàro summability of orthogonal expansion plays an important role in my past investigation of approximation properties and in the open questions here.

I hope some mathematicians will pursue the questions mentioned here and that I will see the proof of some of the conjectures. I am curious to see (but doubt) counter examples to some of the conjectures given in this article.

2. The Jackson inequality on the sphere

For functions on the unit sphere $S^{d-1} = \{\boldsymbol{x} = (x_1, \ldots x_d) : x_1^2 + \cdots x_d^2 = 1\}$ which are in a normed or quasi-normed space B, I introduced in [22] the moduli of smoothness

$$\omega^r(f, t)_B = \sup\{\|\Delta_\rho^r f\|_B : \rho \in SO(d),\ \rho\boldsymbol{x} \cdot \boldsymbol{x} \geq \cos t\} \tag{2.1}$$

where $SO(d)$ is the group of orthogonal matrices whose determinant equals 1, $\Delta_\rho^1 f(\boldsymbol{x}) = \Delta_\rho f(\boldsymbol{x}) = f(\rho\boldsymbol{x}) - f(\boldsymbol{x})$ and $\Delta_\rho^{k+1} f(\boldsymbol{x}) = \Delta_\rho\big(\Delta_\rho^k f(\boldsymbol{x})\big)$ for any integer $k \geq 1$.

The Jackson inequality on B (when proved) is given by

$$E_n(f)_B \leq c\omega^r(f, 1/n)_B \tag{2.2}$$

where

$$E_n(f)_B = \inf\Big\{\|f - \varphi\|_B : \varphi \in \operatorname{span}\Big(\bigcup_{k=0}^{n} H_k\Big)\Big\}, \tag{2.3}$$

$H_k = \{\varphi : \widetilde{\Delta}\varphi = -k(k+d-2)\varphi\}$ (the spherical harmonic polynomial of degree k) and $\widetilde{\Delta}$ is the Laplace–Beltrami operator (the tangential component of the Laplacian Δ). The Hardy space $\mathcal{H}_p(S^{d-1})$ was introduced in [6] and in [7].

Conjecture 2.1. *For $B = L_p(S^{d-1})$, $0 < p < 1$ and for the Hardy space $B = \mathcal{H}_p(S^{d-1})$, $0 < p < 1$, (2.2) is valid.*

For $B = L_p(S^{d-1})$, $1 \leq p \leq \infty$ (2.2) was proved in [23]. For a Banach space B for which $T_\rho f(\boldsymbol{x}) = f(\rho\,\boldsymbol{x})$ are continuous isometries, that is

$$\|T_\rho f\|_B = \|f\|_B \quad \text{and} \quad \|f(\rho\,\cdot) - f(\,\cdot\,)\|_B = o(|\rho - I|) \quad \text{as} \quad |\rho - I| \to 0, \tag{2.4}$$

(2.2) was proved in [13] (see Theorems 3.4 and 6.3) for even $d > 3$ and in [15] for odd $d \geq 3$ (as a result of Theorem 6.1 and Lemma 6.4 there together with [13, Theorem 6.3]). One can observe that the above includes the establishment of (2.2) for the Hardy space $\mathcal{H}_1(S^{d-1})$ and some other non-rearrangement invariant Banach spaces. For $B = L_p(S^{d-1})$, $0 < p < 1$ and $r = 1$ (2.2) was proved in [14]. All these results are partial results of (2.2) and can serve as motivations for Conjecture 2.1.

Conjecture 2.1 is also motivated by its analogues for functions on T or T^d; that is, (2.2) where $E_n(f)_B$ is given by

$$E_n(f)_B = \inf(\|f - \varphi_n\|_B : \varphi_n \text{ a trigonometric polynomial of degree } \leq n) \tag{2.5}$$

and $\omega^r(f, t)_B$ are the classical moduli of smoothness i.e.

$$\omega^r(f, t)_B = \sup_{|\boldsymbol{h}| \leq t} \|\Delta_{\boldsymbol{h}}^r f(\boldsymbol{x})\|_B\,, \quad \Delta_{\boldsymbol{h}}^r f(\boldsymbol{x}) = \sum_{k=0}^{r} \binom{r}{k}(-1)^k f\Big(\boldsymbol{x} + \big(\frac{r}{2} - k\big)\boldsymbol{h}\Big). \tag{2.6}$$

For such $E_n(f)_B$ and $\omega^r(f, t)_B$ when $B = L_p(T)$ or $B = L_p(T^d)$, $0 < p < 1$, (2.2) was proved in [41,42] and [38], and when $B = \mathcal{H}_p(T)$ in [40]. For any Banach space of functions on