Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 232 (2018) 1–5

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Meromorphic tangential approximation on the boundary of closed sets in Riemann surfaces

P.M. Gauthier[a,*], P.V. Paramonov[b], F. Sharifi[c]

[a] *Département de mathématiques et de statistique, Université de Montréal, CP-6128 Centreville, Montréal, H3C3J7, Canada*

[b] *Mechanics and Mathematics Faculty, Moscow State (Lomonosov) University, 119899 Moscow, Russia*

[c] *Department of Mathematics, Middlesex College, The University of Western Ontario, London, Ontario, N6A 5B7, Canada*

Received 13 October 2016; received in revised form 12 April 2018; accepted 12 April 2018

Available online 20 April 2018

Communicated by Doron S. Lubinsky

Abstract

If a closed subset of a Riemann surface is a set of *uniform* meromorphic approximation, then its boundary is shown to be a set of *tangential* meromorphic approximation.

MSC: primary 30E15; secondary 30F99

Keywords: Riemann surface; Continuous analytic capacity; Tangential pole-free approximation

For a closed subset E of a Riemann surface R, we denote by $C(E)$ the algebra of all continuous complex-valued functions on E and by $A(E)$ the subalgebra of those functions in $C(E)$ which are holomorphic in the interior E^o of E. Denote by $M(E)$ the space of functions $E \to \mathbb{C}$ which are uniform limits of meromorphic functions on R which are pole-free on E. Functions in $M(E)$ necessarily lie in the set $A(E)$. We say that E is a set of uniform meromorphic approximation if $A(E) = M(E)$. An outstanding open problem is to determine which closed sets E are sets of uniform meromorphic approximation. A closed set E in R is called a set of

* Corresponding author.

E-mail addresses: gauthier@dms.umontreal.ca (P.M. Gauthier), petr.paramonov@list.ru (P.V. Paramonov), fsharif8@uwo.ca (F. Sharifi).

https://doi.org/10.1016/j.jat.2018.04.009

0021-9045/

tangential (or Carleman) meromorphic approximation, if for every function $f \in A(E)$ and each positive continuous function ϵ on E, there is a function g, meromorphic on R and pole-free on E such that

$$|f(z) - g(z)| < \epsilon(z), \quad z \in E.$$

The following lemma states that these notions are biholomorphically invariant.

Lemma 1. *Let $h : R_1 \to R_2$ be a biholomorphic mapping between Riemann surfaces and let E be a closed subset of R_1. Then E is a set of uniform (respectively tangential) meromorphic approximation in R_1 if and only if $h(E)$ is a set of uniform (respectively tangential) meromorphic approximation in R_2.*

Proof. Suppose $h(E)$ is a set of tangential meromorphic approximation in R_2, that $f \in A(E)$ and that ϵ is a positive continuous function on E. Then, there is a function g meromorphic on R_2 and pole-free on $h(E)$ such that

$$|(f \circ h^{-1})(w) - g(w)| < (\epsilon \circ h^{-1})(w), \quad w \in h(E).$$

Thus, $g \circ h$ is a function meromorphic on R_1, pole-free on E, such that

$$|f(z) - (g \circ h)(z)| < \epsilon(z) \quad z \in E.$$

Hence, E is a set of tangential meromorphic approximation in R_1. The proof in the other direction and for uniform approximation is the same. □

We say a compact set K in a Riemann surface R is a compact parametric disc if there exists an injective holomorphic function $\phi : U \to \mathbb{C}$ defined in a neighbourhood U of K such that $\phi(K)$ is the closed unit disc $\overline{\Delta}$.

For a compact subset K of the Riemann sphere $\overline{\mathbb{C}}$, we denote by $R(K)$ the uniform limits on K of rational functions which are pole-free on K. Considering that $\overline{\mathbb{C}}$ is a Riemann surface, $R(K)$ is the same as $M(K)$.

The following result should be known to specialists, but we could not find an appropriate reference. For the reader's convenience we provide a detailed proof, which is based on deep results of Anatoliĭ G. Vitushkin [11], Mark S. Melnikov [7] and Xavier Tolsa [10].

Theorem 1. *Let K be a compact set in $\overline{\mathbb{C}}$. If $A(K) = R(K)$ then $C(\partial K) = R(\partial K)$.*

Proof. The proof invokes an elementary fact of topology.

Lemma 2. *Let $h : Z \to W$ be a homeomorphism between two topological spaces. If $A \subset Z$, then $\partial(h(A)) = h(\partial A)$.*

Proof. This is an easy consequence of the fact that, for each $X \subset Z$, we have $h(X^0) = (h(X))^0$. □

By Lemmas 1 and 2, we may assume that $K \subset \mathbb{C}$. Let $A(K) = R(K)$. Denote by $\alpha(E)$ the continuous analytic capacity of a bounded Borel set E. By Vitushkin's criterion [11, Th. 1, p. 192], it suffices to prove that there exists $A_1 > 0$ such that for each open disc $B = B(a, r)$ one has $r = \alpha(B) \leq A_1\alpha(B \setminus \partial K)$.

By Tolsa's subadditivity theorem [10], there is a constant $A_2 > 0$ such that, for every pair of bounded Borel sets E and F, one has $\alpha(E \cup F) \leq A_2(\alpha(E) + \alpha(F))$.