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Full length article

Best approximations and moduli of smoothness of functions and their derivatives in L_p, $0 < p < 1$

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Abstract

Several new inequalities for moduli of smoothness and errors of the best approximation of a function and its derivatives in the spaces L_p, $0 < p < 1$, are obtained. For example, it is shown that for any $0 < p < 1$, $k,\ r \in \mathbb{N}$, and a function $f \in W_1^r$

$$\omega_{r+k}(f,\delta)_p \le C(p,k,r)\delta^{r+\frac{1}{p}-1}\left(\int_0^\delta \frac{\omega_k(f^{(r)},t)_p^p}{t^{2-p}}\mathrm{d}t\right)^{\frac{1}{p}}.$$

Similar inequalities are obtained for the Ditzian–Totik moduli of smoothness and the error of the best approximation of functions by trigonometric and algebraic polynomials and splines. As an application, positive results about simultaneous approximation of a function and its derivatives by the mentioned approximation methods in the spaces L_p, $0 < p < 1$, are derived.
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1. Introduction

Let A be a finite interval $[a, b]$ or the unit circle $\mathbb{T} \cong [0, 2\pi)$. As usual, $L_p = L_p(A)$, $0 < p < \infty$, denotes the space of all measurable function f on A such that

$$\|f\|_p = \|f\|_{L_p(A)} = \left(\int_A |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} < \infty$$

and $W_p^r(A)$, $1 \le p \le \infty$, $r \in \mathbb{N}$, denotes the Sobolev space of functions, that is $f \in W_p^r(A)$ if $f^{(k)} \in AC(A)$, $k = 0, \dots, r-1$, (absolutely continuous functions on A) and $f^{(r)} \in L_p(A)$.

Measuring the smoothness of a function by differentiability is too crude for many purposes of analysis. Subtler measurements are provided by moduli of smoothness. Recall that for $f \in L_p$, the classical (non-periodic and periodic) modulus of smoothness of order $r \in \mathbb{N}$ is defined by

$$\omega_r(f, \delta)_p = \omega_r(f, \delta)_{L_p(A)} = \sup_{0<h\le\delta} \|\Delta_h^r f\|_{L_p(A_{rh})},$$

where

$$\Delta_h^r f(x) = \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^\nu f(x + \nu h),$$

$\binom{r}{\nu} = \frac{r(r-1)\dots(r-\nu+1)}{\nu!}$, $\binom{r}{0} = 1$, and $A_{rh} = [a, b - rh]$ in the case $A = [a, b]$ or $A_{rh} = \mathbb{T}$ in the case $A = \mathbb{T}$. We also use the notation $\omega_0(f, \delta)_p = \|f\|_p$.

It is well known (see [4, p. 46, (7.13)]) that for any function $f \in W_p^r(A)$, $1 \le p < \infty$, and $k, r \in \mathbb{Z}_+$

$$\omega_{r+k}(f, \delta)_p \le \delta^r \omega_k(f^{(r)}, \delta)_p. \tag{1.1}$$

It is also possible to estimate $\omega_k(f^{(r)}, \delta)_p$ from above by $\omega_{r+k}(f, \delta)_p$. Such estimate is given by the following weak-type inverse inequality: for $f \in L_p$, $1 \le p < \infty$, and $k, r \in \mathbb{N}$ one has

$$\omega_k(f^{(r)}, \delta)_p \le C_r \int_0^\delta \frac{\omega_{r+k}(f, t)_p}{t^{r+1}} \mathrm{d}t \tag{1.2}$$

(see Johnen and Scherer [14], see also [4, p. 178, Theorem 3.1]). Inequalities (1.1) and (1.2) have important applications in the theory of functions and approximation theory and have been intensively studied in the case of Banach spaces (see, e.g., [1, Ch. 4], [4, Ch. 2 and Ch. 6], and [27]).

In contrast, in the spaces L_p, $0 < p < 1$, there are only some partial results related to (weak) inverse inequalities and some examples of functions for which the classical direct inequalities of type (1.1) are impossible. Thus, Ditzian and Tikhonov [9, Theorem 3.5] proved that for any periodic function $f \in L_p(\mathbb{T})$, $0 < p < 1$, and $k, r \in \mathbb{N}$ one has

$$\omega_k(f^{(r)}, \delta)_{L_p(\mathbb{T})} \le C_{p,k,r} \left(\int_0^\delta \frac{\omega_{r+k}(f, t)_{L_p(\mathbb{T})}^p}{t^{pr+1}} \mathrm{d}t\right)^{\frac{1}{p}}. \tag{1.3}$$

At the same time, it is known that inequality (1.1) is no longer valid for a general f in the case $0 < p < 1$, even if we assume that $f \in C^\infty$ (see [22, Remark 2.6]). Moreover, in the monograph of Petrushev and Popov [25, p. 188], it was mentioned that "*there is no upper estimate of $\omega_k(f, \delta)_p$ by $\omega_{k-1}(f', \delta)_p$ in the case* $0 < p < 1$". Surprisingly, it turns out that such estimation is possible but in terms of weak-type inequalities related to (1.2) and (1.3). Namely,

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