Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 230 (2018) 24–31
Journal of Approximation Theory
www.elsevier.com/locate/jat

Full Length Article

Polynomial approximation on a compact subset of the real line

Vladimir Andrievskii

Department of Mathematical Sciences, Kent State University, Kent, OH 44242, United States

Received 26 January 2018; received in revised form 21 March 2018; accepted 11 April 2018
Available online 19 April 2018

Communicated by Vilmos Totik

Abstract

We prove an analogue of the classical Bernstein theorem concerning the rate of polynomial approximation of piecewise analytic functions on a compact subset of the real line.
© 2018 Elsevier Inc. All rights reserved.

MSC: 30C10; 30E10
Keywords: Polynomial approximation; Green's function; Piecewise analytic function

1. Introduction and the main result

Let $E \subset \mathbb{R}$ be a compact subset of the real line $\mathbb{R}$ and let $\mathbb{P}_n$ be the set of all (real) polynomials of degree at most $n \in \mathbb{N} := \{1, 2, \dots\}$. Also, let for $x_0 \in E$ and $\alpha > 0$,

$$\mathcal{E}_n(|x - x_0|^\alpha, E) := \inf_{p \in \mathbb{P}_n} \sup_{x \in E} | |x - x_0|^\alpha - p(x)|.$$

The starting point of our analysis is the classical Bernstein theory [6,8,7]. According to this theory, for any $x_0 \in (-1, 1)$ and $\alpha > 0$, where α is not an even integer, there exists a finite nonzero limit

$$\lim_{n\to\infty} n^\alpha \mathcal{E}_n(|x - x_0|^\alpha, [-1, 1]) := (1 - x_0^2)^{\alpha/2}\sigma_\alpha.$$

E-mail address: andriyev@math.kent.edu.

https://doi.org/10.1016/j.jat.2018.04.003
0021-9045/© 2018 Elsevier Inc. All rights reserved.

The question as to what happens to the best polynomial approximations for a general set $E \subset \mathbb{R}$ is investigated in monographs [16] and [15, Chapter 10] where the reader can also find a comprehensive survey of this subject.

Now, we consider E to be a set in the complex plane $\mathbb{C}$ and use the notions of potential theory in the plane (see [13,14] for details). Let E be non-polar, i.e., be of positive (logarithmic) capacity $\text{cap}(E) > 0$ and let $g_{\overline{\mathbb{C}}\setminus E}(z) = g_{\overline{\mathbb{C}}\setminus E}(z, \infty)$, $z \in \overline{\mathbb{C}} \setminus E$ be the Green function of $\overline{\mathbb{C}} \setminus E$ with pole at infinity, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the extended complex plane.

Our main objective is to prove the following result.

Theorem 1. *Let $x_0 \in E \subset \mathbb{R}$. If for some $\alpha > 0$,*

$$\limsup_{n\to\infty} n^{\alpha} \mathcal{E}_n(|x - x_0|^{\alpha}, E) > 0, \tag{1.1}$$

then

$$\sup_{z\in\mathbb{C}\setminus E} \frac{g_{\overline{\mathbb{C}}\setminus E}(z)}{|z - x_0|} < \infty. \tag{1.2}$$

Comparing Theorem 1 with [4, Corollary 1] we obtain the following result.

Theorem 2. *Let $x_0 \in E \subset \mathbb{R}$. Then for any $\alpha > 0$, which is not even integer,*

$$\liminf_{n\to\infty} n^{\alpha} \mathcal{E}_n(|x - x_0|^{\alpha}, E) > 0$$

if and only if (1.2) holds.

For the geometry of E satisfying (1.2), we refer the reader to [9,15,10,3] and the many references therein.

2. Auxiliary results

In this section we assume that

$$E = \bigcup_{j=1}^{m} [a_j, b_j], \quad x_0 \in \bigcup_{j=1}^{m} (a_j, b_j) =: \text{Int}(E),$$

where $-1 = a_1 < b_1 < a_2 < \cdots < b_{m-1} < a_m < b_m = 1, m > 1$.

It is known (for example, see [17, pp. 224–226], [5, pp. 409–412] or [2]) that there exists a conformal mapping $w = F(z) = F_E(z)$ of the upper half-plane $\mathbb{H} := \{z : \Im z > 0\}$ onto the domain

$$G = G_E = \{z : 0 < \Re z < \pi, \Im z > 0\} \setminus \cup_{j=1}^{m-1} [u_j, u_j + iv_j],$$

where $0 =: u_0 < u_1 < u_2 < \cdots < u_{m-1} < u_m := \pi$ and $v_j > 0,\ j = 1, \ldots, m - 1$, which can be extended continuously to $\overline{\mathbb{H}}$ satisfying the following boundary correspondence

$$\begin{aligned} &F(\infty) = \infty,\ F((-\infty, -1]) = \{z : \Re z = 0, \Im z \geq 0\}, \\ &F([1, \infty)) = \{z : \Re z = \pi, \Im z \geq 0\}, \\ &F([a_j, b_j]) = [u_{j-1}, u_j], \quad j = 1, \ldots, m, \\ &F([b_j, a_{j+1}]) = [u_j, u_j + iv_j], \quad j = 1, \ldots, m - 1. \end{aligned}$$

Moreover,

$$\begin{aligned} &g_{\overline{\mathbb{C}}\setminus E}(z) = \Im(F(z)), \quad z \in \overline{\mathbb{H}}, \\ &\pi \mu_E([a, b]) = |F([a, b] \cap E)|, \quad [a, b] \subset [-1, 1], \end{aligned}$$

Download English Version:

https://daneshyari.com/en/article/8898430

Download Persian Version:

https://daneshyari.com/article/8898430

Daneshyari.com